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Dynamic output compensator design for time-varying discrete time systems with delayed states

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Abstract— **Convex conditions, proposed as a feasibility test of a set of linear matrix inequalities (LMIs), are given for the design of a partially parameter dependent dynamic output feedback controller for a class of discrete time systems with delayed state. This class concerns the systems with time-varying delay in the state and time varying parameters, both assumed to be available on line. The controller is composed by a classical part, feeding-back the system output and another one where the delayed output of the system is feedback. This last part can be avoided in case of inaccessible delay value, resulting in a more classical control design. Different from other approaches found in the literature, it is used a parameter dependent Lyapunov-Krasovskii functional candidate and a slack matrix variable to reduce the conservatism of the proposed approach. The resulting parameter dependent controller can be obtained on line as a convex combination of some 'vertex' controllers. Numerical examples are presented to illustrated the effectiveness of the proposal.**

I. INTRODUCTION

Systems with delay in the states have been object of intensive study in the last two decades [23]. For both stability analysis and stabilization, the Lyapunov-Krasovskii (L-K) functional approach has found to be the most used [9], [21]. For discrete-time systems, new results have been proposed with the aid of some L-K functional candidates, mainly for robust stability analysis and robust state-feedback synthesis. See, for instance, [4], [5], [7], [16], for non-convex stabilization conditions. In general, L-K functionals are constructed with constant and parameter independent matrices, thus, using quadratic stability approach and usually considering norm-bounded systems [1], [2], [8], [22]. However, this approach can be very conservative specially for time-invariant systems. More new results, like [6], [28] and [11] employ L-K functionals based on quadratic stability. Just a few conditions in the literature deal with polytopic systems, see for instance [15], [17], [18], [12] where convex conditions are provided with parameter dependent L-K functionals. See [20] for the case of swichted systems with state delay.

Despite the large number of studies about systems with delay in the states, there are just a few that present convex methods for synthesis, specially for dynamic output feedback controllers. Chen and Xu [26] present a full-order exponential stable dynamic output feedback controller with $\mathcal{H}_\infty$ criteria for a norm bounded uncertainty system. Dynamic output feedback control is considered by Young *et al.* [27] where a proposal for calculating a dynamic output feedback stabilization controller using static output feedback techniques is presented, only for precisely known systems. Also, He *et al.* [10] present a design for output feedback control for precisely known systems. For time-varying systems, Qiu *et al.* [22] study the problem of delay dependent dynamic output feedback control for a class of uncertain discrete-time switched linear state-delayed systems with $\mathcal{H}_\infty$ guaranteed performance.

In this paper, polytopic time-varying discrete-time systems with time-varying delay in the states are investigated. Both, the parameter of the polytopic representation and the delay are assumed to be available on line. The main contribution of this note is to present convex conditions for designing a dynamic output feedback controller which is dependent on the time-varying parameters. Differently from other approaches found in the literature, the proposed dynamic output controller can be design to cast feedback of both the output signal of the system and its delayed valued. The dynamic controller can be tuned on line assuring the closed loop stability of the system for a wider range of delay variations. This issue is achieved by means of a parameter dependent L-K functional. It is worth to say that the structure of the employed L-K functional is not new, being, in fact, simpler than others such as those in [10], [19], etc. However, it seems that the ideas presented here can be extended to match with more general L-K functional structures. Examples are given to show the effectiveness of the proposal. It is included some time simulations as well as some comparisons with literature results.

Notation: The notation used is quite standard: x_t is the state at time t. $\mathbb{R}$ is the set of real numbers and $\mathbb{N}^*$ stands for the set of the natural numbers excluded the 0. $\mathbf{I}$ and $\mathbf{0}$ are the identity and the null matrices of appropriate dimensions, respectively. $M = \text{block-diag}\{M_1,\ M_2\}$ stands for the block-diagonal matrix M made up by the matrices M_1 and M_2 at the main diagonal. $M > \mathbf{0}$ ($M < \mathbf{0}$) means that M is positive (negative) definite. M' stands for the transpose of M. $\star$ is used to indicate diagonally symmetric blocks in the LMIs. Φ_d denotes the space of discrete vector functions mapping the interval $\mathcal{I}[-d, 0]$ into $\mathbb{R}^n$ with a finite $d \in \mathbb{N}_*$. $\phi_{\hat{t}}^d \in \Phi_d$ denotes a sequence of $d+1$ vectors $x_{\hat{t}}$ with

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$\hat{t} \in \mathcal{I}[t-d, t]$. The j-th term of this sequence is $x_{t+j-1-d} = \phi^d_{t,j} \in \mathbb{R}^n$. It is defined $\|\phi^d_t\|_D = \max_{j\in\mathcal{I}[1,(d+1)]} \|\phi^d_{t,j}\|$ where $\|\cdot\|$ stands for the Euclidean vector norm. Φ^κ_d is the set defined by $\Phi^\kappa_d = \{\phi^d_t : \|\phi^d_t\|_D < \kappa\}$, with $\kappa \in \mathbb{R}_+$. $\hat{\phi}_d$ stands for the null sequence $\hat{\phi}_d = \underbrace{\{\mathbf{0}, \ldots, \mathbf{0}\}}_{(d+1)\text{ terms}}$.

II. Problem statement

Consider the discrete time-varying system with delayed state given by

$$\begin{aligned} x_{k+1} &= A(\alpha_k)x_k + A_d(\alpha_k)x_{k-d_k} + B(\alpha_k)u_k \quad (1)\\ y_k &= Cx_k \quad (2)\end{aligned}$$

with $x_k = x(k) \in \phi^\tau_0(k)$ begin the state vector at k-th sample for $k \in \mathcal{I}[-\bar{d}, 0]$, matrices $A(\alpha_k) \in \mathbb{R}^{n\times n}$, $A_d(\alpha_k) \in \mathbb{R}^{n\times n}$ and $B(\alpha_k) \in \mathbb{R}^{n\times p}$ are convex combinations of vertex matrices A_i, $A_{d,i}$ B_i, $i = 1, \ldots, N$ and $C \in \mathbb{R}^{q\times n}$. $\phi^\tau_0(k)$ is the initial condition, necessary to assure existence and uniqueness for the solutions of (1), with $\tau = \max(d_k)$. x_{k-d_k} is the state vector at d_k past samples and $d_k = d(k) \in \mathbb{N}_*$ is the time-varying delay subject to

$$|d_{k+1} - d_k| = \delta \in \mathcal{I}[0, \bar{d}] \quad (3)$$

$u_k = u(k) \in \mathbb{R}^p$ stands for the control signal, considered given by a dynamic output feedback controller given by

$$\begin{aligned} x_{c,k+1} &= A_c(\alpha_k)x_{c,k} + A_{cd}(\alpha_k)x_{c,k-d_k}\\ &\quad + B_c(\alpha_k)y_k + B_{cd}(\alpha_k)y_{k-d_k} \quad (4)\\ u_k &= C_c x_{ck} + C_{cd}x_{c,k-d_k} + D_c y_k + D_{cd}y_{k-d_k} \quad (5)\end{aligned}$$

$C_c \in \mathbb{R}^{p\times n}$, $C_{cd} \in \mathbb{R}^{p\times n}$, $D_c \in \mathbb{R}^{p\times q}$ and $D_{cd} \in \mathbb{R}^{p\times q}$ are constant and known matrices. The remain matrices of the system and controller are time-varying and given as follows

$$\begin{aligned} &\Xi(\alpha_k) \equiv \\ &\left[\begin{array}{ccccc} A(\alpha_k) & A_d(\alpha_k) & B(\alpha_k) & \mathbf{0} & \mathbf{0} \\ \hline A_c(\alpha_k) & A_{cd}(\alpha_k) & \mathbf{0} & B_c(\alpha_k) & B_{cd}(\alpha_k)\end{array}\right] \\ &= \left[\begin{array}{ccccc} A & A_d & B & \mathbf{0} & \mathbf{0} \\ \hline A_c & A_{cd} & \mathbf{0} & B_c & B_{cd}\end{array}\right](\alpha_k) = \sum_{i=1}^{N} \Xi_i \alpha_{k,i} \quad (6)\end{aligned}$$

with $\Xi_i \in \mathbb{R}^{2n\times 2n+p+2q}$ assembled from the vertex matrices A_i, $A_{d,i}$, B_i, $A_{c,i}$, $A_{cd,i}$, $B_{c,i}$, $B_{cd,i}$, $i = 1, \ldots, N$. The time-varying parameter α_k is supposed available on line and satisfies

$$\Omega \equiv \left\{\alpha_k : \alpha_k \in \mathbb{R}^N, \sum_{i=1}^{N} \alpha_{k,i} = 1, \alpha_{k,i} \geq 0\right\} \quad (7)$$

Thus, $\Xi(\alpha_k)$ belongs to a polytopic domain with vertices Ξ_i, $i = 1, \ldots, N$. By defining an augmented state vector $\zeta_k = \left[\begin{array}{cc} x'_k & x'_{c,k}\end{array}\right]' \in \mathbb{R}^{2n}$, the closed loop system (1)-(5) can be rewritten as

$$\zeta_{k+1} = \mathcal{A}(\alpha_k)\zeta_k + \mathcal{A}_d(\alpha_k)\zeta_{k-d_k} \quad (8)$$

where

$$\mathcal{A}(\alpha_k) = \left[\begin{array}{cc} A(\alpha_k) + B(\alpha_k)D_cC & B(\alpha_k)C_c \\ B_c(\alpha_k)C & A_c(\alpha_k)\end{array}\right] \in \mathbb{R}^{2n\times 2n} \quad (9)$$

and

$$\mathcal{A}_d(\alpha_k) = \left[\begin{array}{cc} A_d(\alpha_k) + B(\alpha_k)D_{cd}C & B(\alpha_k)C_{cd} \\ B_{cd}(\alpha_k)C & A_{cd}(\alpha_k)\end{array}\right] \in \mathbb{R}^{2n\times 2n} \quad (10)$$

Note that the closed loop matrices $\mathcal{A}(\alpha_k)$ and $\mathcal{A}_d(\alpha_k)$ belong to a polytopic domain with vertices given by $\mathcal{A}_i$ and $\mathcal{A}_{d,i}$, $i = 1, \ldots, N$, as done for $\Xi(\alpha_k)$.

If $\zeta_k \in \phi^\tau_t(k) = \hat{\phi}_\tau$ for $k \in \mathcal{I}[t-\tau, t]$, then an equilibrium condition is achieved for the closed loop system (8), since $\zeta_{k+1} = \zeta_k = \mathbf{0}$, $\forall k > t$ and $\forall \alpha \in \Omega$.

Definition 1: For a given $\alpha_k \in \Omega$, the trivial solution of (8) is said uniformly asymptotically stable if for any $\kappa \in \mathbb{R}_+$ such that for all initial conditions $\zeta_k \in \phi^\tau_0(k) \in \Phi^\kappa_\tau$, $k \in \mathcal{I}[-\tau, 0]$, it is verified

$$\lim_{t\to\infty} \phi^\tau_{t,j}(k) = \mathbf{0}, \quad \forall j \in \mathcal{I}[1, \tau+1]$$

The main objective in this work is to formulate convex optimization problems, expressed as LMIs, that can solve the following fundamental issue:

Problem 1 (Compensator design): Given the discrete time-varying system with delayed state (1)-(2), determine, if possible, a dynamic output compensator as in (4)-(5) such that (8) is asymptotically stable.

If the delay d_k is available at each sample time, then it is possible to use a dynamic compensator as described by (4)-(5). It is expected that this degree of freedom can be used to improve the closed loop performance. On the other hand, if the delay d_k is not available, the approach presented in this paper is still valid, with $A_{cd} = \mathbf{0}$, $B_{cd} = \mathbf{0}$, $C_{cd} = \mathbf{0}$ and $D_{cd} = \mathbf{0}$ in (4)-(5).

III. Preliminary results

The following L-K functional candidate is considered in this paper

$$V(\alpha, k) = \sum_{v=1}^{3} V_v(\alpha, k) \quad (11)$$

with

$$\begin{aligned} V_1(\alpha_k, k) &= x'_k \mathcal{P}(\alpha_k)x_k, \quad (12)\\ V_2(\alpha_k, k) &= \sum_{j=k-d_k}^{k-1} x'_j \mathcal{Q}(\alpha_k)x_j, \quad (13)\\ V_3(\alpha_k, k) &= \sum_{\ell=2-\delta}^{1-\underline{d}} \sum_{j=k+\ell-1}^{k-1} x'_j \mathcal{Q}(\alpha_k)x_j \quad (14)\end{aligned}$$

This L-K functional candidate has been used to investigate the stability of (8) in [13], with

$$\mathcal{P}(\alpha_k) = \sum_{i=1}^{N} \alpha_{k,i}\mathcal{P}_i \quad \mathcal{Q}(\alpha_k) = \sum_{i=1}^{N} \alpha_{k,i}\mathcal{Q}_i \quad (15)$$

The following result is well known and will be used in this paper.

Lemma 1 (Finsler's Lemma): Let $\omega \in \mathbb{R}^n$, $\mathcal{R}(\alpha_k) = \mathcal{R}(\alpha_k)' \in \mathbb{R}^n$ and $\mathcal{B}(\alpha_k) \in \mathbb{R}^{m\times n}$ such that $\mathtt{rank}(\mathcal{B}(\alpha_k)) < n$. The following statements are equivalent:

i) $\omega'\mathcal{R}(\alpha_k)\omega < \mathbf{0}, \quad \forall\omega \ : \ \mathcal{B}(\alpha_k)\omega = \mathbf{0}, \quad \omega \neq \mathbf{0}$

ii) $\exists \ \mathcal{X}(\alpha_k) \in \mathbb{R}^{n\times m} \ : \ \mathcal{R}(\alpha_k) + \mathcal{X}(\alpha_k)\mathcal{B}(\alpha_k) + \mathcal{B}(\alpha_k)'\mathcal{X}(\alpha)' < \mathbf{0}$

Proof: The proof follows similar steps of the proof presented in [3] replacing the precisely known matrices by parameter dependent matrices. ■

The following result can be recovered from [14, Th. 2] imposing $G = \mathbf{0}$ and replacing A, A_d, F, P and Q by $\mathcal{A}$, $\mathcal{A}_d$, $\mathcal{F}$, $\mathcal{P}$ and $\mathcal{Q}$, respectively.

Lemma 2 (Stability Analysis): If there exist symmetric positive definite matrices $\mathcal{P}_i \in \mathbb{R}^{2n\times 2n}$, $\mathcal{Q}_i \in \mathbb{R}^{2n\times 2n}$, $i = 1,\ldots,N$, matrix $\mathcal{F} \in \mathbb{R}^{2n\times 2n}$ and a scalar $\beta = 1+\delta$ such that

$$\Theta_1 = \begin{bmatrix} \mathcal{P}_j + \mathcal{F} + \mathcal{F}' & -\mathcal{F}\mathcal{A}_i & -\mathcal{F}\mathcal{A}_{d,i} \\ \star & \beta\mathcal{Q}_i - \mathcal{P}_i & \mathbf{0} \\ \star & \star & -\mathcal{Q}_\ell \end{bmatrix} < \mathbf{0} \qquad i,j,\ell = 1,\ldots,N \tag{16}$$

then, the closed loop system (8) is asymptotically stable $\forall \alpha_k \in \Omega$.

Proof: The positivity of the functional (11)-(15) is assured with the hypothesis of $\mathcal{P}_i = \mathcal{P}_i' > \mathbf{0}$, $\mathcal{Q}_i = \mathcal{Q}_i' > \mathbf{0}$. To (11) be a L-K functional, besides its positivity, it is necessary to verify

$$\Delta V(\alpha_k, k) < 0, \ \forall \ [x_k' \ \ x_{k-d_k}']' \neq \mathbf{0} \tag{17}$$

$\forall \ \alpha_k \in \Omega$. From hereafter, the α_k dependency is omitted in the expressions $V_v(k)$, $v = 1,\ldots,3$, for simplicity of the notation. To calculate (17), consider

$$\Delta V_1(k) = x_{k+1}'\mathcal{P}(\alpha_{k+1})x_{k+1} - x_k'\mathcal{P}(\alpha_k)x_k \tag{18}$$

$$\Delta V_2(k) = x_k'\mathcal{Q}(\alpha_k)x_k - x_{k-d_k}'\mathcal{Q}(\alpha_{k-d_k})x_{k-d_k} + \sum_{i=k+1-d_{k+1}}^{k-1} x_i'\mathcal{Q}(\alpha_i)x_i - \sum_{i=k+1-d_k}^{k-1} x_i'\mathcal{Q}(\alpha_i)x_i \tag{19}$$

and

$$\Delta V_3(k) = (\bar{d} - \underline{d})x_k'\mathcal{Q}(\alpha_k)x_k - \sum_{i=k+1-\bar{d}}^{k-\underline{d}} x_i'\mathcal{Q}(\alpha_i)x_i \tag{20}$$

Observe that, the third term in equation (19), $\Xi_k \equiv \sum_{i=k+1-d_{k+1}}^{k-1} x_i'\mathcal{Q}(\alpha_i)x_i$, can be rewritten as

$$\begin{aligned} \Xi_k &= \sum_{i=k+1-\underline{d}}^{k-1} x_i'\mathcal{Q}(\alpha_i)x_i + \sum_{i=k+1-d_{k+1}}^{k-\underline{d}} x_i'\mathcal{Q}(\alpha_i)x_i \\ &\leq \sum_{i=k+1-d_k}^{k-1} x_i'\mathcal{Q}(\alpha_i)x_i + \sum_{i=k+1-\bar{d}}^{k-\underline{d}} x_i'\mathcal{Q}(\alpha_i)x_i \end{aligned} \tag{21}$$

Using (21) in (19), one gets

$$\Delta V_2(k) \leq x_k'\mathcal{Q}(\alpha_k)x_k - x_{k-d_k}'\mathcal{Q}(\alpha_{k-d_k})x_{k-d_k} + \sum_{i=k+1-\bar{d}}^{k-\underline{d}} x_i'\mathcal{Q}(\alpha_i)x_i \tag{22}$$

So, taking into account (18), (20) and (22), the following upper bound for (17) can be obtained

$$\Delta V(k) \leq x_{k+1}'\mathcal{P}(\alpha_{k+1})x_{k+1} + x_k'[\beta\mathcal{Q}(\alpha_k) - \mathcal{P}(\alpha_k)]x_k - x_{k-d_k}'\mathcal{Q}(\alpha_{k-d_k})x_{k-d_k} < 0 \tag{23}$$

Applying Lemma 1 with $\omega = [x_{k+1}' \ x_k' \ x_{k-d_k}']'$, $\mathcal{B}(\alpha_k) = [\mathbf{I} \ \ -\mathcal{A}(\alpha_k) \ \ -\mathcal{A}_d(\alpha_k)]$, appropriate $\mathcal{R}(\alpha_k)$ and the special choice of $\mathcal{X}(\alpha_k) = [\mathcal{F}' \ \mathbf{0} \ \mathbf{0}]'$ it is obtained

$$\begin{bmatrix} \mathcal{P}(\alpha_{k+1}) + \mathcal{F} + \mathcal{F}' & \mathcal{F}\mathcal{A}(\alpha_k) & \mathcal{F}\mathcal{A}_d(\alpha_k) \\ \star & \beta\mathcal{Q}(\alpha_k) - \mathcal{P}(\alpha_k) & \mathbf{0} \\ \star & \star & -\mathcal{Q}(\alpha_{k-d_k}) \end{bmatrix} < \mathbf{0} \tag{24}$$

which can be recovered from (16) by $\sum_i^N \sum_j^N \sum_\ell^N \alpha_{k+1,j}\alpha_{k,i}\alpha_{k-d_k,\ell}\Theta_1$, $\alpha_k \in \Omega$, completing the proof. ■

In this paper, only dynamic output feedback stabilization design is considered. The results, however, can be extended to other cases including the $\mathcal{H}_\infty$ guaranteed cost control design.

IV. Main Results

This section contains the main result of this paper that is a convex condition for design of a dynamic output feedback control that partially depends on the parameters of the system.

Theorem 1: If there exists matrices Y, X, T, $\bar{P}_{12,i}$, $\bar{Q}_{12,i}$, $\hat{A}_{c,i}$, $\hat{A}_{cd,i}$, and positive definite symmetric matrices $\bar{P}_{11,i}$, $\bar{P}_{22,i}$, $\bar{Q}_{11,i}$, $\bar{Q}_{22,i}$, all belonging to $\mathbb{R}^{n\times n}$, matrices $\hat{B}_{c,i} \in \mathbb{R}^{n\times q}$, $\hat{B}_{cd,i} \in \mathbb{R}^{n\times q}$, $i = 1,\ldots,N$, matrices $\hat{C}_c \in \mathbb{R}^{p\times n}$, $\hat{C}_{cd} \in \mathbb{R}^{p\times n}$, $\hat{D}_c \in \mathbb{R}^{p\times q}$, $\hat{D}_{cd} \in \mathbb{R}^{p\times q}$, and a scalar $\beta = 1+\delta$ such that (30) is verified, then the dynamic output feedback controller (4)-(5) with matrices given by

$$\left.\begin{array}{c} D_c = \hat{D}_c, C_c = (\hat{C}_c - D_cCX)Z^{-1} \\ B_{c,i} = (V')^{-1}(\hat{B}_{c,i} - Y'B_iD_c), \\ A_{c,i} = (V')^{-1}\Gamma_{c,i}Z^{-1} \\ \Gamma_{c,i} = \\ \hat{A}_{c,i} - Y'(A_i + B_iD_cC)X \\ -V'B_{c,i}CX - Y'B_iC_cZ \end{array}\right\} \tag{25}$$

$$\left.\begin{array}{c} D_{cd} = \hat{D}_{cd}, C_{cd} = (\hat{C}_{cd} - D_{cd}CX)Z^{-1} \\ B_{cd,i} = (V')^{-1}(\hat{B}_{cd,i} - Y'B_iD_{cd}), \\ A_{cd,i} = (V')^{-1}\Gamma_{cd,i}Z^{-1} \\ \Gamma_{cd,i} = \\ \hat{A}_{cd,i} - Y'(A_{d,i} + B_iD_{cd}C)X \\ -V'B_{cd,i}CX - Y'B_iC_{cd}Z \end{array}\right\} \tag{26}$$

and matrices N, Z and V satisfying

$$NV = \mathbf{I} - XY \tag{27}$$

$$V'Z = T - Y'X \tag{28}$$

stabilizes asymptotically the discrete-time varying system with delayed state (1)-(2). Besides, (11)-(15) is a L-K functional that assures the asymptotic stability of the closed loop system with

$$\mathcal{P}_i = \Phi' \begin{bmatrix} \bar{P}_{11,i} & \bar{P}_{12,i} \\ \bar{P}'_{12,i} & \bar{P}_{22,i} \end{bmatrix} \Phi; \; \mathcal{Q}_i = \Phi' \begin{bmatrix} \bar{Q}_{11,i} & \bar{Q}_{12,i} \\ \bar{Q}'_{12,i} & \bar{Q}_{22,i} \end{bmatrix} \Phi \tag{29}$$

$$i = 1, \ldots, N \text{ and } \Phi = \begin{bmatrix} \mathbf{I} & -XZ^{-1} \\ \mathbf{0} & Z^{-1} \end{bmatrix}$$

Proof: The proof is inspired in the work of Scherer and co-workers [24] and is based on the stability conditions established in Lemma 2. Note that $\mathcal{F}$ in (16) is a regular matrix due to $\mathcal{P}_j > \mathbf{0}$ and $\mathcal{P}_j + \mathcal{F} + \mathcal{F}' < \mathbf{0}$. Matrix $\mathcal{F}$ and its inverse can be partioned as

$$\mathcal{F} = \begin{bmatrix} Y & M \\ V & \bullet \end{bmatrix}; \; \mathcal{F}^{-1} = \begin{bmatrix} X & N \\ Z & \bullet \end{bmatrix} \tag{31}$$

where $\bullet$ does not matter. Thus, Y and X are nonsingular. By (30) and (29), it is possible to see that $\bar{P}_{\nu\nu,i} > \mathbf{0}$ and $\bar{Q}_{\nu\nu,i} > \mathbf{0}$, $\nu = 1, 2$, $i = 1, \ldots, N$, implying the regularity of Y and X. As done for matrix $\mathcal{F}$, it is possible conclude that

$$\begin{bmatrix} Y' & T \\ \mathbf{I} & X' \end{bmatrix} = \begin{bmatrix} \mathbf{I} & Y' \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{0} & T - Y'X \\ \mathbf{I} & X \end{bmatrix}$$

has full rank and so does $T - Y'X$ (see (28)). Therefore, it is always possible to choose regular matrices Z and V in (28) by an adequate matrix decomposition. LMI (30) can be obtained from (16) by pre- and post-multiplying Θ_1 by $\mathcal{T} = \mathbf{I}_3 \otimes \Lambda'$ and its transpose, respectively, with

$$\Lambda = \begin{bmatrix} \mathbf{I} & X \\ \mathbf{0} & Z \end{bmatrix},$$

taking (29), (31), matrices X, Z, Y, T and N satisfying (27)-(28) and making the following changes of variables: $\hat{D}_c = D_c$, $\hat{D}_{cd} = D_{cd}$, $\hat{C}_c = \hat{D}_c C X + C_c Z$, $\hat{C}_{cd} = \hat{D}_{cd} C X + C_{cd} Z$, $\hat{B}_{c,i} = Y' B_i \hat{D}_c + V' B_{c,i}$, $\hat{B}_{cd,i} = Y' B_i \hat{D}_{cd} + V' B_{cd,i}$, $\hat{A}_{c,i} = Y'(A_i + B_i D_c C) X + Y' B_i C_c Z + V' B_{c,i} X + V' A_{c,i} Z$, $\hat{A}_{cd,i} = Y'(A_{d,i} + B_i D_{cd} C) X + Y' B_i C_{cd} Z + V' B_{cd,i} X + V' A_{cd,i} Z$. ■

When applying Theorem 1, equations (27)-(28) are used to determine V, N and Z with T, X and Y. They can be solved by standard matrix decomposition, such as orthogonal-triangular decomposition (QR), applied on (28) and then isolating N in (27). Obviously, other decompositions yield different values of controller parameters. In this paper, the results does not depend on N. It is expected that this matrix, which is part of $\mathcal{F}^{-1}$, could be exploited in future research, such as for design a controller with some guaranteed perform index. Also, note that conditions on Theorem 1 does not impose any restriction on the L-K functional matrices $\mathcal{P}$ and $\mathcal{Q}$. Thanks to the use of the slack matrix variable $\mathcal{F}$, it is possible to employ a parameter dependent L-K functional candidate, and determine a parameter dependent dynamic output feedback controller. This can be viewed as a counter part of the usually found in the literature, where constant and parameter independent controllers and L-K matrices are employed.

The conditions presented in Theorem 1 can also be used in case of unknown delay. In this case, it is sufficient to impose $\hat{A}_{cd} = \mathbf{0}$, $\hat{B}_{cd}$, $\hat{D}_{cd} = \mathbf{0}$ and $\hat{C}_{cd} = \mathbf{0}$. Thus, the resulting dynamic output feedback controller does not depend on d_k.

Notice that conditions in Theorem 1 can be used to recover the quadratic stability approach, i.e., constant matrices in the L-K functional by imposing $\mathcal{P}_i = \mathcal{P}$ and $\mathcal{Q}_i = \mathcal{Q}$ in (30). This means that (11)-(15) is a constant L-K functional. In this case the LMI presented in (30) must be tested only for $i = 1, \ldots, N$.

It is worth to say that the presented method has two advantages. Firstly, the convex design of a parameter dependent dynamic output feedback controller that can deal with time-varying delay systems. Secondly, the proposed controller structure, that allows the feedback of past values of the output. It is expected that this issue can be exploited in future research, to improve some performance index, such as guaranteed $\mathcal{H}_\infty$ cost.

A. Numerical complexity

The numerical complexity of the proposed conditions depend on the number of variables, $\mathcal{K}$, and on the number of rows in the LMIs, $\mathcal{L}$. Using the program SeDuMi [25], the number of floating point operations performed to solve convex optimization problems has an order given by $\mathcal{K}^2 \mathcal{L}^{5/2} + \mathcal{L}^{7/2}$. Then, the conditions present in Theorem 30 have $\mathcal{K} = 3n^2(2N + 1) + 2n(N + p + q) + 2pq$ scalars variables and $\mathcal{L} = 6N^3 n$ rows.

V. Numerical Examples

In both examples it has been used an Intel Core 2 Duo $T8100$, 2.10 GHz processor with 4 Gb of RAM and the *SeDuMi* [25]. The results achieved are compared with other obtained by conditions available in the literature.

Example 1 (Time-varying system): Consider the unstable discrete time-varying system with matrices

$$A(\rho_k) = A_0(1 + \rho_k); \quad B(\rho_k) = B_0(1 + \rho_k) \tag{32}$$

$$A_d(\rho_k) = A_{d0}(1 + \rho_k); \quad C = [1\,0]; \; D = 0 \tag{33}$$

where the nominal matrices are give by

$$[A_0|A_{d0}|B_0] = \left[\begin{array}{cc|cc|c} 0.9429 & 0.55 & 0.1 & 0.2 & 0 \\ 0 & 1.1 & 0.15 & 0.1 & 1 \end{array}\right]. \tag{34}$$

It is considered that $0 \leq \rho_k \leq 0.2$, which leads to a polytopic representation of this system given by (1) with vertices:

$$[A_1|A_{d1}|B_1] = [A_0|A_{d0}|B_0], \tag{35}$$

$$[A_2|A_{d2}|B_2] = \left[\begin{array}{cc|cc|c} 1.1314 & 0.66 & 0.12 & 0.24 & 0 \\ 0 & 1.32 & 0.18 & 0.12 & 1.2 \end{array}\right]. \tag{36}$$

The purpose here is to design a dynamic output-feedback controller to stabilizing this system. Using Theorem 1 a search has been made on δ for stabilizable controllers. The maximum value achieved is $\delta = 200,939$. For $\delta = 20$ the matrices of the controller are given

$$\begin{bmatrix} \bar{P}_{11,j}+Y+Y' & \bar{P}_{12,j}+T+\mathbf{I} & -Y'A_i-\hat{B}_cC & -\hat{A}_{c,i} & -Y'A_{d,i}-\hat{B}_{cd}C & -\hat{A}_{cd,i} \\ \star & \bar{P}_{22,j}+X+X' & -A_i-B_i\hat{D}_cC & -A_iX-B_i\hat{C}_c & -A_{d,i}-B_i\hat{D}_{cd}C & -A_{d,i}X-B_i\hat{C}_{cd} \\ \star & \star & \beta\bar{Q}_{11,i}-\bar{P}_{11,i} & \beta\bar{Q}_{12,i}-\bar{P}_{12,i} & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & \beta\bar{Q}_{22,i}-\bar{P}_{22,i} & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & \star & -\bar{Q}_{11,\ell} & -\bar{Q}_{12,\ell} \\ \star & \star & \star & \star & \star & -\bar{Q}_{22,\ell} \end{bmatrix} < \mathbf{0}$$

$$i,j,\ell = 1,\ldots,N. \quad (30)$$

by $[C_c|C_{cd}] = \left[\begin{array}{cc|cc} -0.0114 & 0.0239 & -0.0018 & 0.0017 \end{array}\right]$, $[D_c|D_{cd}] = \left[\begin{array}{c|c} -1.4920 & -0.3460 \end{array}\right]$,

$$[A_{c1}|B_{c1}] = \left[\begin{array}{cc|c} -0.7270 & 3.6700 & -210.3141 \\ 0.0001 & 0.0833 & 0.2001 \end{array}\right],$$

$$[A_{c2}|B_{c2}] = \left[\begin{array}{cc|c} -0.8708 & 4.3061 & -252.0757 \\ 0.0002 & 0.0951 & 0.2609 \end{array}\right],$$

$$[A_{cd1}|B_{cd1}] = \left[\begin{array}{cc|c} -0.2434 & 0.2137 & -33.6412 \\ 0.0004 & 0.0077 & 0.0538 \end{array}\right],$$

$$[A_{cd2}|B_{cd2}] = \left[\begin{array}{cc|c} -0.2922 & 0.2640 & -40.3933 \\ 0.0005 & 0.0077 & 0.0697 \end{array}\right].$$

These two controller vertices are used to define a discrete-time parameter varying controller. A time simulation is presented in Figure 1 where the closed-loop system output (top) is shown with the respective control signal (bottom). In this simulation it has been considered a time-varying behavior for α_k as presented in the bottom of Figure 2 and that d_k has a dominant increasing behavior (top of Figure 2). Observe that, d_k attends (3). It is assumed as initial condition $x_k = [1 \quad -1]'$, $k \le 0$. A state perturbation has been applied at $k = 201$, by adding $[-1 \quad 1]'$ to the state vector. At this instant the delay value is $d_{201} = 75$. The consequence of this perturbation appears at different some instants ahead as a combined effect of the increasing delay and the stabilization time of the closed loop system.

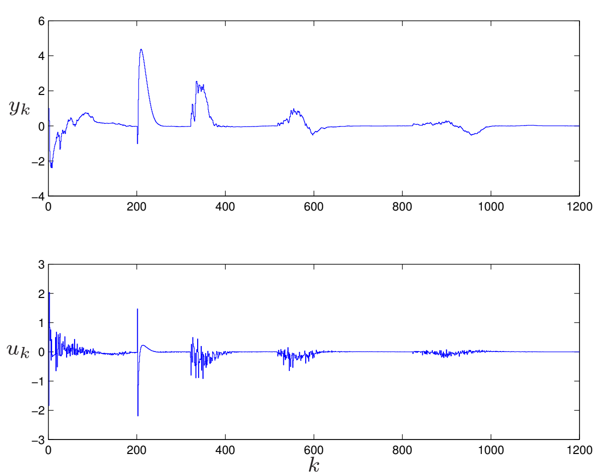

Fig. 1. Closed-loop output signal, y_k on top, and control signal, u_k on bottom.

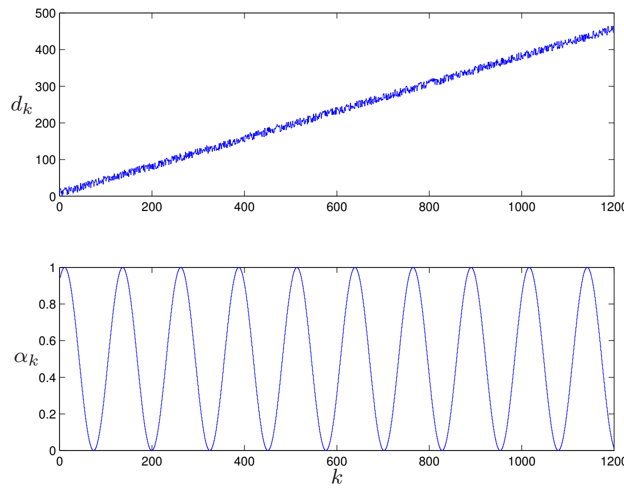

Fig. 2. Time-varying parameters d_k (top) and α_k (bottom).

Such behavior can be observed in Figure 1 at instants about $k = 322$, $k = 518$ and $k = 823$ where the delay is about (see Figure 2) $d_{322} = 120$, $d_{518} = 196$, $d_{823} = 306$, respectively. As can be viewed in Figure 1, the closed loop system is stable. It has been verified that, although stable for large values of delay, the required time to stabilize the closed loop system can be larger than expected for real world applications. Thus, to get a better time response, it is interesting to consider some performance specifications, such as $\mathcal{H}_\infty$ or pole location [17]. In case of unavailable delay value on line, Theorem 1 can still be employed to search a stabilizing dynamic output controller. In this case, a maximum of $\delta = 9$ can be achieved with $A_{cd,i} = \mathbf{0}$, $B_{cd,i} = \mathbf{0}$, $D_{cd} = \mathbf{0}$, $C_c = \left[\begin{array}{cc} -0.0040 & 0.0003 \end{array}\right]$, $D_c = \left[\begin{array}{c} -2.3196 \end{array}\right]$ and

$$[A_{c1}|B_{c1}] = \left[\begin{array}{cc|c} -1.0038 & 0.1511 & -1043.8203 \\ 0.0000 & 0.0021 & 0.0478 \end{array}\right],$$

$$[A_{c2}|B_{c2}] = \left[\begin{array}{cc|c} -1.2038 & 0.1426 & -1252.0126 \\ 0.0000 & 0.0031 & 0.0465 \end{array}\right].$$

Example 2 (Precisely known system): This example is borrowed from [10] where a precisely known system investigated with system matrices

$$\left[\begin{array}{c|c|c} A & A_d & B \\ \hline C & C_d & D \end{array}\right] = \left[\begin{array}{cc|cc|c} 0.9 & 0.5 & 0.3 & 0 & 1 \\ 0.8 & 0.1 & 0.8 & 0.5 & 0.5 \\ \hline 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

and the output signal is given by $y_k = Cx_k + C_d x_{k-d_k}$. Here, the delayed part of the output is omitted, i.e., C_d is made null. In [10] the conditions are delay-dependent and the dynamic controller output has constant matrices and employs only the output signal, i.e. it does not take into account the delayed output signal as done here. Here, a linear search has been made over δ to determine its maximum value such as Theorem 1 is feasible. It has been found $\delta_{\max} = 5508$ (see (3)), for any finite delay. By using the delay-dependent and nonlinear techniques presented in [10] the maximum delay stabilizable is given by $d_{\max} = 1000$. Thus, it is clear that the strategy presented here can lead to less conservative results despite the simpler Lyapunov-Krasovskii functional employed. This wider range of delay variation is reached thanks to the proposed structure of dynamic feedback control, that includes a delayed output signal.

VI. CONCLUSIONS

A convex condition for the design of a partially parameter dependent dynamic output feedback to stabilze discrete time-varying systems with time-varying delay in the state is given in this note. It is used a paramenter dependent Lyapunov-Krasovskii functional and some slack matrix variables to derive a controller which output is based on the current system output and on the delayed output of the system. A numerical example is presented to illustrate the effectiveness of the proposal. It is expected that the approach presented here could be extended to coupe with more complete Lyapunov-Krasovskii functionals, leading to less conservative stabilization conditions. Also, the presented approach can be used to take into account some performance index such as the $\mathcal{H}_\infty$ guaranteed cost.

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