Snow on the Mountains:

Number in Grocheio in the thirteenth century

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Abstract

Grocheio's treatise *Ars Musice* occupies the central folios in the manuscript Harley 281 (GB-Lbl Harl. 281), principally copied by a single scribe around 1300 AD. It is replete with numbers and Hindu-Arabic numerals and thereby is totally distinguished from all the other treatises on music theory in the manuscript. Grocheio's work, apparently not well known in medieval times, much better known now since the work of Rohloff, Seay and Page *inter alia*, is preserved in only two manuscripts (the Harley and Darmstadt D-DS 2663). In this paper we discuss Grocheio's surprisingly frequent use of number, and the role of the numerals in the manuscript in their contexts. We also investigate what evidence this provides the width of Grocheio's knowledge of mathematics and science.

Harley 281 (GB-Lbl Harl. 281) is a remarkably coherent collection of medieval treatises on music theory arranged in what appears to be a chronological sequence.[3] The manuscript is essentially the work of a single scribe.[4] In the present paper we are principally concerned with the treatise by Johannes de Grocheio and the use made therein of Hindu-Arabic and other numerals and numbers.

It is generally believed that Grocheio was active in the last part of the thirteenth century, and the references in his text indicate that he was in Paris. His work was edited early last century by Rohloff (1930, 1943, and 1972). Subsequently a translation was made by Seay (1967). More recently there has been considerable discussion of Grocheio's work from the perspective of music theory; see for example, Christopher Page (1993a, 1993b). In particular, in Page (1993a: 131), we find:

'It would make a fascinating study to trace the incursion of arabic numerals into the study of musica.'

Grocheio's scientific context can be inferred, not only from what we know of Paris at that time, but also from evidence within the manuscript. At the time of Grocheio, which is thought to be in the mid- to late thirteenth century, Hindu-Arabic numerals had been introduced into Europe but they were not in common use until about the end of the fourteenth century. (See Smith and Karpinski (1911), especially pp. 132-133 and Cajori (1928: 34, section 54).) In Grocheio's work such numerals are present on every page except the first. This contrasts strongly with all the other treatises in the Harley manuscript, where very few numbers occur. Moreover, in these other treatises, these numbers are always presented in Roman numerals or Latin words. On the other hand, while several of the other treatises in Harley 281 give examples in musical notation, that of Grocheio gives no such examples, it only mentions the names of the various chants (*cantus*) and other musical works that he gives as musical examples. The other surviving manuscript of Grocheio's treatise is Darmstadt (D-DS 2663, ff. 56r–69r). This latter is in a collection of ecclesiastical works from the library of the Carthusians of St. Barbara at Cologne. Unlike the Harley manuscript, this includes few works on music. Grocheio's treatise demonstrates a continuing preoccupation with number and, with the few exceptions noted below, numbers in the Harley manuscript are always presented using Hindu-Arabic numerals or Latin words.[5] In the Darmstadt manuscript *all* numbers are represented in Hindu-Arabic numerals or in Latin words. There are no occurrences of Roman numerals in D-DS 2663 at all. Of the 14 folios (39r-52r) of Harley 281 that Grocheio's work occupies, only one page (f. 39r) does not contain a number or numeral, and this page is the introduction to the work. Most pages contain several numbers. Indeed there is a total of 150 numerals, an average of more than five per page.[6] Grocheio's treatise on music is therefore unique in the Harley manuscript because of its concern with number. Its pervasive use of Hindu-Arabic numerals in both manuscripts is extremely striking. First we consider the presentation in the extant manuscripts, next Grocheio's background, his knowledge and sources, and finally the use of numbers and numerals.

The two manuscript versions of Grocheio's work are clearly written by different scribes. The Harley manuscript, which measures 220mm by 145mm, is written on parchment, not of high quality, and shows considerable signs of use, especially in the final work by Guy of Saint-Denis (see van der Klundert, 1988). The whole text of Harley 281 is written in a single, neat but not beautiful, carefully done, professional scribal hand, suggesting it was for student use. There are emendations in at most two other hands, one of them a scholar's hand. The ink is uneven, making it easy to see where the scribe's quill was recharged. The Latin is careful and accurate. It is of note that there are very few corrections in Grocheio's treatise. There is, however, one erasure in the Grocheio text (f. 39v, line 17), which requires an insertion. This is of a numerical nature and we discuss it below. In the manuscript of the treatise of Guy of Saint-Denis, the last treatise in the manuscript, there is extensive crossing out and emendation, but there is very little emendation of any kind elsewhere in the whole Harley manuscript. All of this leads us to believe that the manuscript was intended as a workbook.[7] The Darmstadt manuscript is significantly smaller than the Harley one, measuring 160mm by 120mm. Its scribe writes with a far finer, and much smaller, hand in two columns: presumably to aid legibility because of the smallness of the hand. The ink is much more even than in the Harley manuscript. Again there are very few corrections.

On the whole the actual texts are remarkably close with only minimal variant readings. Such variants involve at most two or three words at a time. Nevertheless these small differences are rather significant and we discuss them below. In the Darmstadt manuscript the only numerals are Hindu-Arabic ones: there are no Roman numerals at all (but there are Latin words for numbers). These numerals are written with great confidence and consistency, and demonstrate that the scribe was adept at writing them. In particular the 8s are very beautifully written.

The contents of Harley 281 (based on the entry in [Harley catalogue] 1808) are as follows:

f.1r-4v	Miscellaneous introductory material including, 1. Lines on solmisation, 2. Lines on the effects of the eclipses of the sun, 3. A Guidonian solmisation hand, 4. A near contemporary contents list, and 5. Acrostic verses around the name GUIDO.
f.5r-16v	Guido of Arezzo, *Micrologus* (after 1026)
f.16v-21r	Guido of Arezzo, *Regulae rhythmicae* (*c*.1026)
f.21r-22v	Guido of Arezzo, *Prologus in Antiphonarium* (*c*.1026)
f.22v-24v	Guido of Arezzo, *Epistola ad Michaelem* (opening extract only, *c*. 1030)
f.24v-25v	*Dialogus de Musica*
f.25v-32r	Pseudo-Odo, *Dialogus de Musica* (*c*. 1000)
f.32r	Interpolation "Quantumcumque vero omnium … neumas et concordes constituunt" probably unique and possibly by Guy de Saint Denis
f.32r-33v	Guido of Arezzo, *Epistola ad Michaelem* (closing extract)
f.33v-34r	Eighteenlines of verse
f.34r-38v	St Bernard Tonary (12th century)
f.39r-52r	Johannes de Grocheio, *Ars musice*[8]
f.52v-58r	Petrus de Cruce, *Tractatus de tonis* (early 14th century)
f.58v-96v	Guy de Saint-Denis, *Tractatus de tonis*

In the old table of contents ostensibly in a near contemporary hand on f. 4r of the Harley manuscript, we read:

f. 35 [now relabelled as f.39r] *Tractatus musice a quodam innominato.*[9]

Grocheio's name does not occur in Harley 281 at all. The only place in which it is known to occur is in a colophon, which identifies the author of this treatise, added to the Darmstadt manuscript on f. 69r in a different hand, where we read:

explicit theoria magistri iohannes de grocheio. regens // Parisius

Here it seems clear that the *regens* has been added after the preceding words (as stated in the notes in the Hessisches Universitäts und Landesbibliothek) and *Parisius* was added even later.

At the outset it is important to distinguish between what is actually written on the parchment of the manuscripts and the ideas presented. We particularly distinguish between the numerals – the representation of numbers by means of the digits (in modern orthography 0, 1, 2, 3, 4, 5, 6, 7, 8, 9) used to write numbers in the Hindu-Arabic form or by Roman numerals – and the abstract idea of numbers. To clarify the distinction consider the numeral 2 written here in this paper as opposed to the non-zero solution of the equation $x + x = x$ times x, which is an abstract number. Of course, the number, which is the solution, is written as represented (viz. denoted) by the numeral 2. We shall first treat the numerals that occur in the manuscript and subsequently the abstract ideas of number and other scientific ideas.

The Hindu-Arabic digits, which are used to write numerals, and which gradually made their way into the Latin West (in a process that is still largely undocumented), are quite distinct from Roman letters. They are not taken from any alphabet; they are new symbols akin to pictures or diagrams.[10] These numerals made their way only slowly from India to the West through Arab lands, hence the appellation 'Hindu-Arabic'. Coming from Asia, they were discussed in a lost work of al-Khwarizmi (ninth century),[11] and in the Latin West the processes involved in using them (addition, multiplication, etc.) became known as *algorism*, derived from al-Khwarizmi's name. Finally the numerals came through the Maghreb into Europe. The transmission through Europe seems to have begun in Spain and Italy. It gave rise to major changes in thinking. As Rashed (1992) says,[12] 'L'histoire de l'arithmétique arabe dans la civilisation latine, n'est ... pas moins que l'histoire de l'interaction entre deux civilisations.'

There are two major, and quite distinct, contributions from the new system. First there is the conceptual change from the additive system of Roman numerals to the place notation of Hindu-Arabic numerals, which seems to have been very difficult to understand. Secondly there are the processes of algorism, where operations are recorded permanently on a surface such as parchment or paper, rather than disappearing after the calculation as they do with an abacus. It certainly took a long time for algorism and Hindu-Arabic numerals to be universally accepted. Indeed there was very strong resistance, as has been noted by Menninger (1969, especially 424-31).

We do not need to discuss the processes of algorism here since such processes do not occur in our text, so we shall restrict ourselves solely to the representation of numbers. There were two novelties in the very way of writing down numbers using the new approach. On the one hand there were the actual symbols used, the digits or 'figures',[13] .9.8.7.6.5.4.3.2.1. and .0., and on the other the system of decimal-place notation, where moving to the left means multiplying by 10. Hindu-Arabic figures had been used by Gerbert around 1000 AD for labelling the calculi in his abacus, but he did not use the place notation. What was strange to practitioners in the thirteenth century about the use of the numerals was the way that one figure could mean different things depending on how far to the left it occurred. Thus ‘3’ might mean ‘three’ or ‘thirty’ or ‘three thousand’, while the basic Roman numbers such as ‘I’, ‘C’ could only affect numbers adjacent to them as in ‘IX’ or ‘XI’ or ‘MCM’. Hindu-Arabic numerals appear therefore to have been regarded as having a nature different from letters and seem, originally, to have been treated in the same way as pictures or diagrams, rather than as letters from an alphabet. Acceptance of this quite revolutionary system was slow and spasmodic but was greatly facilitated by the popular accounts of Sacrobosco and Villedieu.[14]

In Rohloff's 1972 edition of Grocheio’s text all the Hindu-Arabic numerals found in the Darmstadt manuscript have been normalized to Roman numerals although Rohloff also consulted the Harley manuscript. Of course this is a traditional transcription convention. In the version of the Harley text in the Thesaurus Musicarum Latinarum, the numerals in the Harley manuscript, both Roman and Hindu-Arabic, have been transcribed as Hindu-Arabic numerals in the Wolf (1899-1900) transcript and entirely as upper case Roman letters in the Rohloff (1943). On the other hand, in the online transcription at http://www.uga.edu/~thema/ (accessed 6 September 2005) the original orthography has been preserved and the Roman numerals vi and vii occur thus in lower case (the latter twice). Rohloff in his 1972 edition consistently uses upper-case letters, whereas the few Roman numerals in the Harley manuscript are always in lower-case and although there are no Roman numerals in the Darmstadt one Rohloff transcribes the Hindu-Arabic ones into upper case Roman numerals. Likewise the transcribers of the two versions in Thesaurus Musicarum Latinarum were, of course, applying standard conventions, but this convention significantly obscures what happens in the manuscripts. However, since Hindu-Arabic numerals were still not fully accepted in Grocheio’s time, the presentation of the numerals and the shift from Roman to Hindu-Arabic numerals is a significant aspect of the manuscript tradition. We discuss this further below.

We have used the word ‘figure’ interchangeably with ‘digit’. It is easy nowadays to make a distinction between *figura* in the musical sense, and ‘figure’ as meaning a digit. The distinctions are not, however, entirely clear in Grocheio, but he carries the idea of using novel signs to denote, in his case, musical notes, into his discussion of musical notation for the *longa* in [18.3], R 171, H. f. 46r, D f. 63r:[16]

And they invented certain general signs and indeterminate figures in order to represent sound, by which they were not able to represent cantus or sound adequately. And so others added a prescription. For they placed one square figure, having a line descending or ascending straight from the right-hand part, which they called a long, and they distinguished it in a twofold [*way*]*: a perfect and an imperfect long.*

The word *figura*, as used here, refers to musical notation. The use of the English word ‘figure’, in the sense of a digit or a symbol for a numeral between 0 and 9, appears only to have come into use with the advent of algorism, thereby replacing the Greek word *schema* (whence our English word 'scheme'), around Grocheio's time (see Onions 1966: 355).

In the Harley and Darmstadt manuscripts the Hindu-Arabic digits used are (written in their modern forms) 1, 2, 3, 4, 5, 6, 7, 8, 9 but 0 was not used. The omission of zero does not seem to be significant since no number that would require a zero is mentioned or even referred to in the text. However, numbers requiring place notation are used: specifically 12 (frequently), and more significantly 256 and 243 ([4.13], R 53, H f. 41r, D 58r). The actual forms of the Hindu-Arabic numerals used in Harley 281 are the standard Western Arabic forms (see Hill 1915), which are virtually identical to our modern ones apart from 4, 5 and 7. Both cardinal and ordinal numbers are used: cardinal numbers are the counting numbers 1, 2, 3, ... and ordinals are ordering numbers, usually written first, second, third, ... or 1st, 2nd, 3rd, ... in English (although we often employ the unadorned numbers 1, 2, 3, ... in dates, reading them as 'first', 'second', 'third',...). On occasion Grocheio distinguishes cardinals from ordinals. The Harley scribe appears to have pronounced the Hindu-Arabic numerals in Latin. For example, he uses Hindu-Arabic numerals in combinations to be read in Latin, e.g. '4' is used as an ordinal number in the form '.4.us' (in [20.5], R 193, H f. 47r, D f. 64r), for 'quartus' and '3' (as a cardinal number) in '.3.bus' for 'tribus' in [4.11], R 50, H f. 41r, D f. 58r. The representation of the cardinals and ordinals leads us to believe that, in the extract below, where we find '.4.or' and '.5.o', the former was read '*quattuor*' (cardinal 4) and the latter was read '*quinto*' (ordinal 5).[17] Thus we have in [26.7], R 231, H. f. 49r, D. 65v (where we have retained these numeric forms):

Et quamquam ut uidetur quilibet tonus plures possit habere inceptiones infra terminos sue latitudinis: determinatas tamen vnicuique attribuunt. puta primo .5o. que scilicet in .d. et .E. grauibus et .e. f. a. acutis. Secundo vero .4or. scilicet in .e. graui et .d. et .e. et .f. acutis. Tercio .6. scilicet in .d. e. f. et .g. grauibus et in .a. et .C. acutis. Quarto vero .5. scilicet in .c. d. e. f. g. Quinto .4. scilicet in .f. graui. et .b. rotundo. et .b. c. quadratus. Sexto vero .3. scilicet in .d. e. et .f. graui. Septimo .4. scilicet in .g. graui. et .b. c. d. acutis. Octauo .4. scilicet in .f. et .g. grauibus et .a. et .c. acutis.

This explicit distinction is lost in the English transcriptions cited above (where we have simply '4' and '5'):

And although, as is apparent, any tone can have many beginnings under the limits of its range; yet they assign fixed [*limits*] *to each, that is, 5 to the first* [*tone*]*, namely on d- or E-*gravis *or e-, f-, a-*acutis. *Indeed 4 to the second, namely on e-*gravis *or d-, or e- or f-*acutis. *6 to the third, namely on d-, e-, f- or g-*gravis *and on a- or C-*acutis. *Indeed 5 to the fourth, namely on c, d, e, f, g. 4 to the fifth, namely on f-*gravis *or b-*rotundus *or b-* [*or*] *c-*quadratus. *Indeed 3 to the sixth, namely on d-, e- or f-*gravis. *4 to the seventh, namely on g-*gravis *and b-, c-, d-*acutis. *4 to the eighth, namely on f- and g-*gravis *or a- or c-*acutis.

Here Grocheio is preparing the discussion about how his 8 modes may be differentiated from each other. He continues to call them 'by a common name and numeral, namely first, second, third etc.' (f. 49r) and he introduces ordinal numbers spelled out as words 'primus, secundus, tercius' (*ibid*.). He continues by dividing these eight into odd ones and even ones, calling the odd ones 'principal, authentic and masculine' (*ibid*.) and the even ones 'by opposing differential characteristics' (i.e. their opposites) (*ibid*.). In the discussion of naming the modes, which follows, the manuscript moves comfortably between ordinal and cardinal systems as well as between Roman and Greek number systems, for example:

[26.3], R 227, H. f. 49r, D. f. 65v *vnde primus dicitur autenticus protus idest primus auctorizatus. Secundus vero plaga proti. Tertius autenticus deuterus idest secundo auctorizatus. Quartus plaga deuteri. Quintus autenticus tritus idest tercius auctorizatus. Sextus plaga triti. Septimus autenticus tetrardus idest quartus auctorizatus. Octauus vero plaga tetrardi.*

Whence, the first is called the authentic protus*, that is, the first according to authority. The second, indeed,* [*is*] *the plagal of the* protus*. The third* [*is*] *authentic* deuterus*, that is, the second according to authority. The fourth* [*is*] *the plagal of the* deuterus*. The fifth* [*is*] *authentic* tritus*, that is, the third according to authority. The sixth* [*is*] *the plagal of the* tritus*. The seventh* [*is*] *authentic* tetrardus*, that is, the fourth according to authority. The eighth, indeed,* [*is*] *the plagal of the* tetrardus.

The use of Hindu-Arabic numbers in Grocheio's treatise as presented within the Harley manuscript is unpredictable. The copyist would appear to use Hindu-Arabic numerals for simplicity when he writes of a proportion as '.256. ad .243.' ([4.13], R 53, H f. 41r, D 58r). However, in the discussion of the number of consonances a range of practices appears. 'Three' is spelt out as 'tres' in reference to the three consonances and to the three perfections in sounds ([3.3], R 23, H f. 40r, D f. 57r), as it is again in the analogy, treated below, of the mother as diapason, the daughter as diapente and a third proceeding from them. Indeed the Harley manuscript quite often uses the Latin words for numbers, but occasionally Roman numerals are employed. Thus we find five occurrences of the Hindu-Arabic figure 7, and shortly thereafter the Roman numeral *.vii*. twice, once for the number of planets (actually written as *stelle*) and the once for the number of principles in human art:

[4.8], R 45, H f. 40v D 57v *Others, however, reduce them all to 7. ... These people, however, take the source of their saying from the sayings of the poets... saying that there are 7 gifts of the spirit and 7 planets in heaven and 7 days in a week, by the multiple repetition of which the whole year is measured. And similarly they say there are seven concordances in sound.*[18]

[4.9], R 47-8, H ff. 40v-41r, D f. 57v *For the vii stars, with their forces, have sufficed for the diversity of generation and decay of the whole universe. And therefore, it was reasonable to posit the vii principles in human art, which are the causes of the diversity of all the sounds.*[19]

The novelty of writing figures is evident in the differences that emerge between the two manuscript versions of Grocheio's work in their use of numerals. There are differences of representation, there is a gap or erasure and there are varying practices within the Harley manuscript. We treat these in turn.

The Darmstadt manuscript uses Hindu-Arabic numerals throughout; there are no Roman numerals at all. In the Harley manuscript there are a very few occurrences of Roman numerals. As noted above, we have *.v. species* on f.40r, *.vii. stelle* on f. 40v and *humana .vii. principia* on f. 41r, but, with one further exception, Harley then uses Hindu-Arabic ones consistently thereafter. The exception, on f. 39v, is noteworthy. Here the Roman numeral *.vi.* is used at the second occurrence of a numeral in the written text but within the context of stating the proportion 12 to 6. Thus we find:

[1.3], H f. 39v *Invenit unum in dupla proportione ad alterum sicut sunt .12. ad .vi.*

[1.3], R 15 D f.56v *Invenit unum in dupla proportione ad alterum sicut sunt .12. ad .6.*

Three sentences later Harley has an erasure that has not been filled in, and this is despite the fact that there are some corrections (but not affecting numerals) in the Harley manuscript. At this point the Darmstadt manuscript incorrectly repeats .12. ad .6., the previous proportion to be found in the previous sentence on folio 56v. In fact the correct proportion would be .12. ad .8., the proportion for the diapente.[21] This immediately raises the question of the understanding of the relevant scribes. Did the Harley scribe discover a mistake in his source and erase what he had written, but yet remain reluctant or unable to insert the correct proportion? The Darmstadt scribe appears either to have copied what was incorrect and not to have questioned what he was writing, or else, having encountered an omission, to have inserted what he felt should be there. He is certainly very confident with Hindu-Arabic numerals, so if he did make a mistake it was surely because of his lack of technical knowledge about proportions in music and not of Hindu-Arabic numerals *per se*. The question therefore arises as to whether there was an error in the original. The consistency of the two texts is extremely high. There is only one notable exception. This is the misreading *nix in montibus* (snow on the mountains), which is correctly rendered in Harley as *aux in motibus*, an astronomical reference. We shall discuss Grocheio's, as opposed to the scribe's, knowledge in this area, below. However, the fact that this is a misreading strengthens our view that the Darmstadt scribe did not understand the technicalities of what he was copying even though he did handle Hindu-Arabic numerals with ease.

Here are two conjectures as to why the numerals were recorded in these different ways. First, perhaps the Darmstadt scribe did not understand the technical context and just copied what was written in his exemplar without trying to understand them: in the first case, the ratio 12 to 6 and, in the second, the astronomical reference. This is supported by the fact that the writing of the figures in that manuscript is confident and does not disturb the flow of the text. Moreover, the Darmstadt scribe significantly, but understandably, misread an astronomical reference, which we treat below, again revealing a lack of scientific knowledge. Further the Harley scribe demonstrated that something was wrong by making the erasure, since the correct text should have been '.12. ad .8.'. It seems clear that Darmstadt scribe did not have this correct version to hand. Another possibility, which appears more plausible for the text in the Harley manuscript than for the Darmstadt one, is that the manuscript was dictated, and that the scribe was completely at ease, writing down either the Latin word for the number or the Roman numeral or the Hindu-Arabic numeral, depending on which way he envisaged it. It is difficult now to establish which of these possibilities is the more likely. What is clear, however, is that Hindu-Arabic numerals were still not the standard, even when the Harley manuscript was written.[22]

We next turn to Grocheio's intellectual, and in particular, scientific, environment. Grocheio refers to music as performed in Paris on three pages (H f. 42r, 44r, 46v). *Musica* was part of the quadrivium being taught in the University of Paris. Indeed *musica*, understood as the speculative theory rather than the practice of music, was the focus of intense study in thirteenth-century Paris. The most significant musical theorists from about 1240 to 1350 such as Hieronymus de Moravia (*Tractatus de musica*); Johannes de Garlandia (*De mensurabili musica*); Lambertus (*Tractatus de musica*); Franco of Cologne (*Ars cantus mensurabilis*); St Emmeram Anonymous; Anonymous 4; Jacobus of Liège (*Speculum musice*), all worked at, or had contact with, Parisian institutions. The theory of music, along with *arithmetica*, geometry and astronomy, could not remain unaffected by scientific debate. We prefer to use the Latin word, *musica*, to 'music' since it principally connotes music theory. In the same way as we opted for *musica*, we prefer the Latin word, *arithmetica*, over the modern English 'arithmetic' because it carries with it the legacy of Boethian arithmetic, which is very different from present day arithmetic. It should be borne in mind that Grocheio was very much concerned with the practice of music, in particular in Paris, and he cites both vernacular and liturgical examples.

Grocheio's discussion of *musica* is based on the *De instituitione musica* of Boethius but he does make certain challenges to Boethian orthodoxy. Pythagorean number theory, as transmitted to the Latin West and handed down by Boethius, had developed from the discovery that the ratios in music could be expressed numerically. Boethius's understanding of the foundation of music was profoundly Pythagorean, since it was based on the whole-number ratios involved in producing harmonious intervals of sound. Boethius retells the tale derived from Nicomachus (see Levin 1994) of Pythagoras passing a blacksmith's shop and hearing hammers of different weights striking consonant and dissonant intervals. The story was well known to early scholars, and is recorded by

Boethius (*op.cit*., Book I, Chapter VI). An extreme version of the associated doctrine affirms that the physical world is a material working out of numerical truth, and that this truth is demonstrated in the musical consonances produced through the proportions between the numbers (see Guthrie and Fideler 1987). This had led to the claim that 'the whole heaven is a harmonia and a number' and to the notion of the harmony of the spheres. It produced the belief that 24, the number of notes on the aulos, 'equals that of the whole choir of heaven'.

The influences current in Paris in the area of *musica* and *arithmetica* at the time of Grocheio had another major source in addition to Boethius, namely Aristotle. Islamic culture had absorbed much Greek philosophy, and had brought Aristotle to a new prominence. However the work of Boethius in both *musica* and *arithmetica* remained highly influential. His treatises on both disciplines have much material in common and both depend heavily on the arithmetic of Nicomachus (see D'Ooge 1926). Grocheio was clearly trained in the Boethian tradition and refers to Boethius directly six times, but on the other hand he refers to Aristotle seven times by name. So Grocheio's theory develops very differently from that of Boethius. Without always abandoning the old views, he often modifies them, especially in the light of experience, for example reflecting Aristotle's view (see *Metaphysics* 987b, 985b32-986a2, 1090a20-23, 1093a28-b4; *On the Heavens*, 290b21-3)· that the ideas such as those mentioned above about the notes on the aulos and that of the music of the spheres are fallacies proceeding from Pythagoras's mistaken fundamental idea that 'numbers are the causes of the reality other things'.

Towards the end of the treatise (on H f. 49r, D f. 64v), Grocheio identifies music, computus (the way of calculating, for example, the date of Easter), and grammar as three disciplines essential for any churchman.[23] The privileged place he gives to number in this treatise suggests that he may also have taught *arithmetica*. This would be consistent with what Beaujouan (1954) has observed: that *arithmetica*, although not on the official syllabus, was still taught in an informal fashion in Paris. (See also Denifle and Chatelain 1964, 1:78, no. 20.)

The new way of writing numbers, using the Hindu-Arabic figures, and the art of algorism, facilitated large calculations. For example those that are required in the construction of astronomical tables or measuring the distance of the sun. Works, such as those of al-Farabi (d. 950) and al-Kindi (d. *c*. 870), had recently come into Spain, and such works were translated by Dominicus Gundissalinus and, in particular by Gerard of Cremona (1114-1187) and Iohannes de Sacrobosco (John of Holywood, 1195-1256). Both the Arabic version of Ptolemy's *Almagest*, (see Thorndike 1949 and Toomer 1984) and Galen's *Tegni*,[24] are among the 79 works translated from Arabic by Gerard in Toledo (see Lemay 1978). However a direct Latin translation (the Salernitan translation) from the Greek was also available although it had a limited circulation.

Hindu-Arabic numerals became known largely through the popular algorism of Sacrobosco and the rhymed one of Alexander de Villedieu (written 1202, Villedieu d.

1240, see Steele 1922).[25] However, only a limited number of people became adept with the new methods of algorism. Even in the fifteenth century the need for knowledge of algorism in Paris was not essential for many clerics. Thorndike (1944: 225-226 translated from Vat. Pal. Lat. 1252, Commendation of the clerk: an educational treatise, chapter 8) quotes '... it is customary in the education of schoolboys to combine the practice of music with the milk of grammar, and also the art of the algorism which serves practical arithmetic. ... But for him who turns the sacred page or who seeks canon law it is enough to have the trivium in order to comprehend the eloquence of divine scriptures.'

Next we consider Grocheio's awareness of mathematics and abstract ideas. This far exceeds what was necessary for *musica*. On the very first page, [0.4], R 6, H f. 39r, D f.56r, he mentions the distance of the sun.

For just as one seeing the method of finding the distance of the body of the sun from the centre of the earth will not be made confused but will be made knowledgeable, so one seeing the discovery of principles of music will be more disposed to knowing.[26]

Knowledge of this is very likely to have come from the *Almagest*, which, as we have noted, was available in Paris, possibly in Gerard of Cremona's translation. Sacrobosco's *De Sphera* does not mention the distance of the sun. The *Almagest* contains a clear diagram in Book VII, but it requires the accompanying text to understand the diagram.[27]

Further evidence for Grocheio's awareness of astronomy comes from his use of the phrase *aux in motibus* on f.41r of the Harley manuscript. In the Darmstadt manuscript on f. 58r this is rendered as *nix in montibus*: 'snow on the mountains'. It is easy to sympathize with the Darmstadt scribe's reading since Rohloff (1972) made the same error, though he did note that the Harley manuscript reads *aux in motibus*, which is correct. The context and translation are as follows:

Tonus autem multipliciter dicitur uelud aux in motibus. vno enim modo dicitur [*de*] *eleuatione depressione, et fine cantus ...*

Now a tone is spoken of in many ways, just like the apogee in [*planetary*] *motions, for it may be spoken of by the rise, the fall and the end of cantus ...*

Aux, augis, is the transcription of the Arabic word *awj* meaning 'apogee'.[28] 'The part of the eccentric [circle in which a planet revolves] which has the greatest distance from the center of the world is called aux, or the greatest distance.' (Pedersen 1974: 452.) So here Grocheio, who, with Aristotle, rejects the idea of the music of the spheres ([5.6], H f. 41v, D f. 59r), is neverthess comparing the motion of the apogee of a planet's motion with the way the musical sound, in a cantus in a particular tone, goes up and down.

There are three principal places whence Grocheio could have become aware of *aux* and *motus*. One is from the *De Sphera* of Sacrobosco, where both occur (in close proximity) in the fourth and final chapter (see Thorndike 1944). A second is from the *Almagest* but this was a very difficult and long work (see Toomer 1984). The third possible source was the *Theorica planetarum,* which is often attributed to Gerard of Cremona (see Pedersen 1974, 451-465).[29] This work is about the motions of the planets, sun and moon but, unlike other works, such as that of Campanus of Novara (see Benjamin and Toomer 1972), bearing the same title, does not discuss the construction of a planetarium. Perhaps *Theorica planetarum* is also the work referred to by Grocheio in [5.6], R 67, H f. 41v, D f.58v: *Cuius ymaginatio et possibilitas debet tradi in libro de theoria planetarum.* ([*W*]*hose ideas and hypothesis ought to be followed in the book concerning the theory of the planets* [*theoria planetarum*].) This attribution has one problem, since the 'whose' refers to Aristotle. On the other hand the reference cannot be to Aristotle's *De Caelo.* It is also worth noting that Ptolemy is mentioned elsewhere in Harley 281. This is in the *Alius prologus* to the *Micrologus* of Guido of Arezzo. On f. 5r we read:

Hoc autem ab inde percepi ubi tholomeus. Duo inquit cromonis capita utriusque phebi collateralia mediocris armonie dulcedine ducta, difficili habencium inquistione siderum mercurio mediante . mirabili meloditate asserimus resonare.

Grocheio's mention of two specifically arithmetic facts reveals his knowledge of mathematics. First, Grocheio is, unsurprisingly, aware of 8 as a cubic number but he also writes of 'proportions between two cubes' on f. 40r. We discuss this below where we turn to proportions as opposed to numbers. Secondly, on [7.7], R 87, H f. 42v, D f.59v, he mentions perfect numbers, that is to say, numbers such as 6 and 28, where all the divisors add up to the original number. Thus the first perfect number is $6 = 1 + 2 + 3$ and the second is $28 = 1 + 2 + 4 + 7 + 14 = 28$.

A number of six elements is the first in the genus of perfect numbers.

This, however, is common knowledge from Boethius.

There was a continuing debate over primacy between numbers and proportions, but it was a debate into which Grocheio chose not to enter.[30] For the Pythagoreans number took precedence and this was endorsed by both Plato (*Republic*, 526c) and Aristotle (*Metaphysics* i.2): *For those which involve fewer principles are more exact than those which involve additional principles*, e.g. arithmetic rather than geometry. Other quantities, including proportions, lines, planes, and solids were treated separately from numbers (see Grattan-Guinness 1996) although Aristotle seemed to claim that there were universal proofs (see *Posterior Analytics* i.5, and *cf. Metaphysics* xiii.2). Moreover, the theory of ratio for magnitudes in Euclid, Elements Book V is completely separate from the treatment of ratio for number in Elements Book VII and parts of Book VIII, although the proofs in Book V would be applicable from a modern point of view. We find

Grocheio considering the (philosophical) foundations concerning proportion and number. Some contemporaries felt that proportion came from nature and preceded number, while others held that numbers were fundamental and that proportion came from them. Grocheio refuses to determine which is the more fundamental:

[2.6], R 26, H f. 40r, D f. 56v *All these* [*Plato and Boethius and their followers*] *have taken this foundation for their position in this, that proportion, as they say, is found firstly and in itself in numbers and is attributed through numbers to other things. But this foundation is not certain for the disciples of Aristotle. For they would say that proportion is perhaps first among prime qualities and natural forms if an utterance is assigned in order to signify this. It does not belong to this work to consider who of these may be speaking the truth, but to where the first principles of the sciences are considered.*[31]

Proportions also occur in the discussion of the ditone. In practical terms it was easy to create a proportion of, say, 9:8 on a monochord, thereby producing a tone. But if one wanted to produce a wider interval, it was quite practical to do this in stages by using simple superparticular (see below) or other proportions, and then repeating them. In terms of the arithmetic, however, Hindu-Arabic numerals facilitated the description. Thus on f. 41r we find:

[4.14], R 54, H f. 41r, D f.58r *Further, a ditone is a concordance containing 2 tones which, compared to the preceding sound, is seen to be proportional as 81 is to 64.*[32]

This comes immediately from repeating the (superparticular) proportion: 8 units plus one unit to 8 units, twice, successively yielding a proportion 9:8 times 9:8 = 81:64. Surely no one would attempt to construct the latter proportion directly.

As was traditional, Grocheio sometimes uses arguments based on analogy; here is one that involves analogy with the number of proportions. Since the world was seen to reflect the spiritual sphere, the use of analogy was a standard method for arguing why there should be a certain structure to it. This is evident in the passage quoted above from ff. 40v-41r concerning the seven stars and the seven human principles. However, Grocheio constantly refers to his own experience, particularly his experience of music as performed in Paris. In his discussion on the number of consonances, his approach is so based (see especially [2.4-2.9], R 32ff.,?? H f. 40r, D f. 57r) because, for Grocheio, observation often supersedes various arguments by analogy. There are two basic questions for Grocheio. On the one hand, how many consonances are there? And on the other, why should there be precisely that many? He agrees with the views of some others that there are five such consonances, and then questions why, if the number of consonances is based, analogically, on the number of kinds of proportion, there is not the same number of consonances as there are kinds of proportion, namely five. In order to follow his argument, we need to consider various kinds of proportion.

Boethius lists five types of proportion, deriving this from the work of Nicomachus (see D'Ooge 1926). Proportions, otherwise known as relative quantities, are thought of through measuring one (length) against another. Only whole multiples of the basic measure are considered. However, because one measure may be a multiple of another, in the same way that a metre is a hundred times a centimetre, any proportion can be expressed in many different ways, such as 1:2 or 10:20 or 7:14. For Nicomachus, multiples come first. In modern algebraic notation, mn: m. Next come superparticular proportions. These are proportions of the form[33] $(x + 1){:}x$, for example, 3:2 or 4:3 or 9:8. There is one extra measure (or aliquot part) beyond one whole. Nowadays we would often express this as a fraction: thus 3/2 or 4/3 or 9/8. A superpartient proportion is a proportion where we have a whole plus more than one extra measure,[34] for example 12:8 or 7:5. Here 12 is 8 (that is to say, one whole) plus 4 extra measures. When we have more than one whole we have a multiple superparticular, as in 17:8, which is of the form[35] (2 times 8) +1:8 or we have a multiple superpartient, as in 15:4, which is of the form[36] (3 times 4)+3:4.

Grocheio also writes of proportions between two cubes (H f. 40r, D f.57r):

And Plato the scholar wished to demonstrate natural things through mathematics; in Plato's book entitled the Timaeus[37] *he has stated the number of the elements, since between two cubes, two intermediate proportions are always found.*

Here we have a sequence of proportions. Consider two cubes (or cubic numbers), for example, 8 = 2 times 2 times 2 and 27 = 3 times 3 times 3. Between them are the numbers 12 and 18 which lie in equal proportions, that is to say, the proportions between adjacent numbers in the list 8, 12, 18, 27 are always 2:3. There are always two such numbers between any two cubes. If the cubes are a^3 and b^3, then these numbers can be written as a^2b and ab^2 and the ratio between adjoining terms will always be $a{:}b$.

Grocheio's argument is not short but we paraphrase it as follows:

From [2.4-2.8] H f. 40v, D f. 57v: *Some say that consonances are infinite in number, others that there are three, arguing by numbers following Pythagoras and Nicomachus the arithmetician. Plato said in the Timaeus that between two cubes two proportions are always found. Boethius followed this in his De Musica.*

All said proportion is found in number and number is found in everything.

However, if proportion comes from numbers they do not show the cause of the number of consonances.[38]

Eventually he concludes (in [3.1], R36, f. 40r), that there are indeed only three consonances, but he is still dissatisfied ([2.10], R33-5, H f. 40r, D f.57r), because he cannot find a justification: he says that, from his observations, it 'seems difficult to define the reason for the number of consonances.'

[2.10], R33-5. H f. 40r, D f. 57r: *Further, if a consonance be natural, it can be recognized by its end. That which is natural is best shown from its end, as Aristotle says in the second book of the Physics: 'For initially the end moves the efficient force and finally completes the work.' If, indeed, we are dealing with music, knowledge of it is sufficient through form. Therefore, because of these things, and of many others, it seems difficult to define the reason for the number of consonances.*[39]

On other occasions he does use a traditional argument by analogy. However, here again we find that number is central. It is unsurprising that Grocheio should use such an argument. He is clearly cognizant of his predecessors, and for them analogy was a favourite way of understanding the world. Thus in [4.8 R45-46], f. 40v, (which we quoted in full above) he focuses on 7 gifts of the spirit and 7 planets in heaven and 7 days in a week. In [3.3], in concluding his discussion on the number of consonances, he invokes the Trinity. Here he manages to combine both analogy and experience saying: *And perhaps just as He is in the glorious Trinity, so in a certain way He teaches through this experience*. He then continues with an analogy with the Trinity.

[3.3], R38-39, H f. 40v, D 57v *For there is one first harmony, like a mother, which is called the diapason by the ancients, and another, like a daughter, contained in it, called the diapente, and a third proceeding from them which is named the diatessaron and these three, sounded at the same time, give the most perfect consonance. And perhaps certain Pythagoreans led by a natural inclination sensed this, not having dared, however, to express it in such words but speaking of it in numbers through metaphor.*[40]

Here we have the analogy of the perfection of the Trinity being reflected in the three consonances.

We have shown that Grocheio had considerable knowledge of mathematics and science, far beyond Boethius and also beyond what was necessary for the teaching of *musica* in Paris. We have seen that, throughout his treatise on the theory of music, Grocheio is constantly concerned with number. Such a pervasive use of number indicates a facility that is quite distinctive. The influence of the actual Hindu-Arabic numerals is, however, quite subtle. They are prevalent in the two manuscripts that we have. The small but significant differences between the manuscripts suggest there might have been an earlier exemplar, but it is also possible that the Darmstadt manuscript was copied from the Harley, though the reverse seems impossible because of the misreading of *aux in motibus*. The closeness of the actual texts, and the fact that, with very few exceptions, Hindu-Arabic numerals are used exclusively throughout both manuscripts, suggests that there was a common source. Further Grocheio was conscious of the way that Hindu-Arabic numerals allow one to write down any number, however large, using only a few symbols (viz. the digits 0 to 9), in the same way as the grammarian can write down any saying using only the small number of letters in the alphabet. He writes in [7.2], R 173, Harley f. 46r, D 63r:

Et quemadmodum gramaticus ex paucis litteris earum coniunctione et situatione potest dictionem quamlibet designare. Et artificialiter numerans ex paucis figuris earum prepositione et postpositione numerum quemlibet infinitum designare. Ita musicus ex tribus figuris cantum quelibet mensuratum.

And just as the grammarian is able to represent any word with a few letters by their conjunction and placement, and someone numbering by artifice [*can*] *represent any unbounded number with a few figures by placing them before and after, so a musicus* [*can*] *represent any measured cantus from the three figures.*

It is therefore hard to imagine that the source of the use of Hindu-Arabic numerals was anyone other than Grocheio himself.

There is a significant question as to what contribution the scribes may have made in putting down Grocheio's words – and numbers – on parchment. The Harley scribe uses both Roman and Hindu-Arabic numerals, so presumably he was familiar with both; the Darmstadt one uses only Hindu-Arabic numerals but with great confidence, which would be consistent with his using them frequently. On the other hand the Harley scribe seems to have been better informed about the content of the text, even being aware of the novel word *aux*. Nevertheless, the written text of the treatise, in both manuscripts, demonstrates a facility with numbers that is quite remarkable. In her article (1997), Gillian Evans wrote an excellent analysis of the change from the abacus to algorism from which Page (1993a: 131) quotes: 'it may be no coincidence that the numerical bases of musical notation did the same [i.e. took a new turn at this time].' Given Grocheio's continual inclusion of numbers on every page bar one, it would not be at all surprising if he had indeed used Hindu-Arabic numerals, a still relatively recent innovation in the study of the sciences.

Much more work is needed into the transcription of numbers within music treatises of this period, and indeed of the general uptake of the new Hindu-Arabic numerals and algorism in the twelfth and thirteenth centuries. We believe we have at least begun to pursue the task suggested by Page, of studying the incursion of Hindu-Arabic numerals into the study of *musica*.

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References

Beaujouan, G. [1954] 1991. 'L'enseignement de l'arithmétique élémentaire à l'Université de Paris aux XIIIe et XIVe siècles' reprinted in *Par raison de nombres, : L'art Du Calcul Et Les Savoirs Scientifiques Médiévaux*, *Variorum Collected Studies Series ; CS 344.* Aldershot: Variorum; Brookfield, VT: Gower, XI, 98.

Benjamin, B. and G.J. Toomer 1972. *Campanus of Novara & Medieval Planetary Theory: "Theorica Planetarum"*, Medieval Science Series; No. 16. Madison, WI: University of Wisconsin Press.

Burnett, C. 2005 'The semantics of Indian numerals in Arabic, Greek and Latin', *Journal of Indian Philosophy*, 33 (in press), 2005, read at the IIAS Workshop - Asian Contributions to the Formation of Modern Science, The Emergence of Artificial Languages, Workshop on September 20-21, 2002, Leiden, The Netherlands.

Cajori, F. [1928] 1974. *A history of mathematical notations*. La Salle, IL: Open Court Pub. Co..

J. N. Crossley and A. S. Henry, 'Thus spake al-Khwarizmi', Historia Mathematica 17 (1990), 103-131.

Denifle, H. and A. Chatelain, eds 1964 [1891-99]. *Chartularium Universitatis Parisiensis* // sub auspiciis Consilii Generalis Facultatum Parisiensium ex diversis bibliothecis tabulariisque collegit et cum authenticis chartis contulit Henricus Danifle ; auxiliante Aemilio Chatelain**.** 4 vols. Paris, reprinted Bruxelles: Culture et civilisation.

Evans, G. 1997. 'From Abacus to Algorism: Theory and Practice in Medieval Arithmetic', *British Journal for the History of Science*, 10 (1977), 114-131.

E. Grant, ed. *A source book in Medieval Science*. Cambridge, MA: Harvard University Press.

Grattan-Guinness, I. 1996 'Numbers, Magnitudes, Ratios and Proportions in Euclid's Elements: How did He Handle Them?', *Historia Mathematica*, 23:355-375.

Guthrie, K.S. and D.R. Fideler 1987. *The Pythagorean Sourcebook and Library: An Anthology of Ancient Writings Which Relate to Pythagoras and Pythagorean Philosophy.* Grand Rapids, MI: Phanes Press.

[Harley Catalogue] 1808. *A Catalogue of the Harleian manuscripts, in the British Museum*, with indexes of persons, places and matters, vol.1, London: By command of His Majesty King George III.

Hill, G.F. 1915 *The Development of Arabic Numerals in Europe* . Oxford: Oxford University Press.

Jowett, B. [1892] Plato *Timaeus*, tr.. Champaign, IL: Project Gutenberg, earlier published in vol. 3 of *The Dialogues of Plato* tr. into English with Analyses and Introductions by B. Jowett, M.A. in Five Volumes. 3rd edition revised and corrected. Oxford: Oxford University Press.

Lemay, R. 1978. Gerard of Cremona. In *Dictionary of Sscientific Biography*, vol. XV (Supplement I) Roger Adams – Ludwik Zejszner: topical essays, C.C. Gillispie, ed. in chief, 173-192. New York: Charles Scribner's Sons.

Levin, F.R. tr. & comm. 1994 . *The Manual of Harmonics of Nicomachus the Pythagorean.* Grand Rapids, MI: Phanes Press.

van der Klundert, S. ed & stud. 1988 *Tractatus de tonis, Guido von Saint-Denis*. Bubenreuth: Hurricane Publishers.

Menninger, K. 1969. *Number words and number symbols; a cultural history of numbers*, tr. Paul Broneer from the rev. German ed. Cambridge, MA: MIT Press.

Menso Folkerts (1997): *Die älteste lateinische Schrift über das indische Rechnen nach al-Hwarizmi. Edition, Uebersetzung und Kommentar*, Bayerische Akademie der Wissenschaften, Philosophisch-historische Klasse, Abhandlungen, neue Folge, Heft 113 (Munich, 1997),

Onions, C.T. ed 1966. *The Oxford Dictionary of English Etymology* with the assistance of G.W.S. Friedrichsen and R.W. Burchfield. Oxford: Clarendon Press.

D'Ooge, M.L. [1926] 1972. *Nicomachus of Gerasa, Introduction to arithmetic.* Translated into English by Martin Luther D'Ooge, with studies in Greek arithmetic by Frank Egleston Robbins and Louis Charles Karpinski. New York: Macmillan, reprinted New York: Johnson Reprint Corp..

Page, C. 1993a. *Discarding Images: Reflections on Music and Culture in Medieval France*. Oxford: Clarendon Press; New York: Oxford University Press.

Page, C. 1993b. Johannes de Grocheio on Secular Music: A Corrected Text and a New Translation’, *Plainsong and Medieval Music*, 2:452-71.

Pedersen, O. 1974. Anonymous: The Theory of Planets, tr. (annotated by E. Grant). In E. Grant (1974) 451-465.
Rashed, R. 1992. Preface to André Allard, *Muhammad Ibn Al-Khwarizmi. Le calcul indien (algorismus). Versions latines du XIIe siècle*, unnumbered page. Leuven: Peeters.

Rohlоff, E. ed 1972. *Die Quellenhandschriften zum Musiktraktat des Johannes de Grocheo. In Faksimile herausgegeben nebst Übertragung des Textes und Übersetzung in Deutsche, dazu Bericht, Literaturschau, Tabellen und Indices*: Leipzig: Deutscher Verlag für Musik.

Rohloff, E. 1930. *Studien zum Musiktraktat des Johannes de Grocheo*. Leipzig.

Rohloff, E. 1943. *Der Musiktraktat des Johannes de Grocheo nach den Quellen Neu Herausgegeben mit Übersetzung, Deutsche und Revisionsbericht*. Series *Media Latinitas Musica*, vol.2, 41-67. Leipzig: Gebrüder Reinecke. Transcribed as http://theme.music.indiana.edu/tml/14th/GRODEM_TEXT.html, accessed 10 October 2005.

Seay, A. 1967. *Johannes de Grocheo Concerning Music*, Colorado College Music Press Texts/Translations 1. Colorado Springs: Colorado College Music Press.

Smith, D.E. and L.C. Karpinski 1911. *The Hindu-Arabic Numerals*. Boston, MA & London: Ginn.

Steele, R. 1922 *The Earliest Arithmetics in English*, Early English Text Society, Extra series 118. London: Early English Text Society.Thorndike, L. 1944. *University Life and Records in the Middle Ages*. New York: Columbia University Press.

Thorndike, L. 1949. *The Sphere of Sacrobosco and its commentators*. Chicago: University of Chicago Press.

Toomer, G. J. tr. & ann. 1984. *Ptolemy’s Almagest*. London : Duckworth.

Wolf, J. 1899-1900. "Die Musiklehre des Johannes de Grocheo," Sammelbände der Internationalen Musikgesellschaft 1 (1899-1900): 69-120. . Transcribed as http://theme.music.indiana.edu/tml/14th/GROTHE_TEXT.html, accessed 10 October 2005.

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3 The other manuscripts are in chronological order and therefore, as we do not know Grocheio's dates exactly, we are assuming that his work is included in the correct chronological order.

4 We shall discuss the whole Harley manuscript in a forthcoming paper by a group ... [*details omitted to preserve anonymity*]. The Harley manuscript catalogue ([Harley catalogue] 1808) says "c.1315-1318" but we believe it may be at least a decade earlier.

5 'Numerals' are the representations of numbers using either the Roman form, e.g. xii, or the Hindu-Arabic form, e.g. 12. Hindu-Arabic numerals are formed from the digits 0, 1, 2, ... , 9. 'Numbers' are the abstract entities.

6 There should be 152, as there are in the Darmstadt manuscript, but there is an erasure which we note below.

7 This was originally suggested to us by Charles Burnett, private communication, April 2005.

8 In the [Harley catalogue] (1808) it says *De Musica*.

9 The last word is missing in the transcription in the [Harley catalogue] (1808) p.104.

10 Burnett has independently noted this in his 2005 writing 'The retention of the visual order of the Indian numerals, I think, adds to the evidence for their being conceived as different from written text, ... Numerals were *hors de texte*, and like pictures, or geometrical diagrams, they kept the same directionality as they passed from one language context to another.'

11 The full text has recently been discovered by Folkerts (1997), but see also Crossley and Henry (1990).

12 'The history of Arabic arithmetic in Latin civilisation is nothing less than the history of the interaction between two civilisations.'

13 This was the standard order and followed the Arabic way of writing. It is perhaps worth noting that the *direction* in which numbers were written down with the digit signifying the largest power of ten on the extreme left, was the opposite to the natural Arabic way of writing from right to left. This is because the Arabs had taken over the Indian system of numerals, which followed the Indian direction of writing. It was only later that people began to write them in the present day order: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Moreover the Hindu-Arabic numerals always occur with a punctus at each side of the numeral.

14 There were many copies of Alexandre de Villedieu, *Carmen de algorismo*, and John Sacrobosco, *De arte numerandi*, circulating in the thirteenth century and indeed they were rendered into English very early. See Steele (1922).

16 Our quotations are from a new translation of Grocheio's work currently being prepared by a team [names temporarily omitted to preserve anonymity]. Numerical references in square brackets in this paper are to our numbering in this translation followed by the Rohloff (1972) page number and then the Harley and Darmstadt folios numbers.

17 In fact the suffices are written directly above the digits.

18 *Alii autem omnes ad .7. reducunt. ... Isti autem ex dictis poetarum originem sui dicti capiunt. et cum hoc rationes probabiles adducunt. dicentes esse .7. dona spiritus. et in celo .7. planetas et in septimana .7. dies quibus multociens resumptio totus annus mensuratur. Et similiter in sonis esse concordantias septem dicunt.*

19 *Ad diversitatem autem generationum et corruptionum totius universi .vii. stelle cum earum virtutibus suffecerunt. Et ideo rationabile fuit ponere in arte <41 r> humana .vii. principia que omnium diuersitatum sonorum cum armonia cause essent[.] que quidem cause concordantie appellantur.*

21 Page (1993a), 172, appears to err in his description of Grocheio's use of Roman versus Hindu-Arabic numerals. Moreover, in Rohloff's transcription we find: *Invenit unum in dupla proportione ad alterum sicut sunt .xii. ad .vi*.

22 Hindu-Arabic numerals were used for calculating but mixtures of Roman and Hindu-Arabic numerals, together with Latin words, were often used for the dates that occurred at the end of texts.

23 Sacrobosco's *Algorism* was, according to Grant (1974):94 was the most popular algorism circulating at that time. Moreover, the three extant works of Sacrobosco, his *Algorism*, *De Sphera* and his *Computus*, are often found together (see Thorndike 1949:3). In addition, the *Theorica Planetarum*, which we mention below, is also found adjoining these works.

24 It should be noted that on Grocheio also mentions the *Tegni* of Galen, which was also translated by Gerard of Cremona. On f. 40v, he writes: *We, in fact, do not intend to expound their differences here, nor*

to go into all the details, but to hand down universal rules, according to our ability, just as the universal rules of the art of medicine are handed down in the book of Galen, which is called Tegni.

25 Such numerals are also to be found in the tables in the manuscript of Gerard of Cremona's translation of Ptolemy's *Almagest* in Melbourne, State Library of Victoria manuscript F 091 P95A. However the numbers occurring in the main text in that manuscript are all written out in Latin words.

26 *Sicut enim videns modum inueniendi distanciam corporis solis a centro terre non admirabitur sed factus erit sciens. Sic videns inuentionem principiorum musice magis erit dispositus ad sciendum.*

27 See Toomer (1984). Toomer actually translated the Almagest from Greek but the diagram reproduced in Toomer is very close indeed to that in Gerard's translation in the Melbourne manuscript (see the previous note but one). Indeed Toomer states that he has tried not to modernize the ancient diagrams.

28 We are grateful to Charles Burnett for clarifying this point. Personal communication, April 2005.

29 Pedersen (1974: 451) questions the authorship of this work, but Lemay (1978, see especially p.189), only a few years later, is positive about Gerard of Cremona being the author. The oldest manuscript is from 'perhaps a little before 1300' (Pedersen, *ibid.)*.

It is interesting to note that the two words that Pedersen leaves in the original Latin are *aux* and *motus*.

30 The issue has been brushed aside in modern mathematics by conflating numbers and ratios. For a detailed presentation see Grattan-Guinness (1996) where he carefully distinguishes Euclid's concepts of quantities,: numbers, geometrical magnitudes, and ratios. The theory of proportions is concerned with comparisons of ratios (see especially p.361).

31 *Omnes autem isti fundamentum sue positionis accipiunt in hoc quod proportio ut dicunt primo et per se in numeris inuenitur et per numeros est aliis attributa. Sed istud fundamentum apud discipulos aristoteles non est certum. Dicerent enim forte proportionem primo esse inter primas qualitates et formas naturales si uox sit imposita ad hoc signandum. Quis autem istorum verum dicat non est huius negocii pertractare sed ubi prima principia scienciarum pertractantur.*

32 [4.14] *Dytonus autem est concordantia continens .2. tonos que sono precedenti comparata sic proportionari videtur sicut .81. ad .64.*

33 This can also be expressed as $(1 + 1/x)$:1.

34 This can also be expressed as $(1 + n/x)$:1, where n is strictly between 1 and x.

35 This can also be expressed as $(m + 1/x)$:1, where m is greater than 1.

36 This can also be expressed as $(m + n/x)$:1, where m is greater than 1, and n is strictly between 1 and x.

37 "Solid bodies are always compacted not by one mean but by two," in Jowett's translation (Jowett 1882).

38 [2.10 R33] *Adhuc autem si consonancia sit naturalis, ex fine cognosci habet. Naturalis enim potius ex fine demonstrat. ut ait aristoteles. secundo physicorum. finis enim primo movet efficientem et ultimo complet opus. Si vero musica. eius cognitio sufficiens est per formam. Propter hec itaque et propter talia plura difficile videtur assignare propter quid de numero consonanciarum.*

39 See previous footnote.

40 *Et forte sicut est in trinitate gloriosa. Ita quodam modo in hac experiencia docet. Est enim una prima armonia quasi mater, que dyapason ab antiquis dicta est. Et alia quasi filia in ista contenta dyapente dicta. Et tercia ab eis procedens que dyatessaron appellatur. Et iste tres simul ordinate consonanciam perfectissimam reddunt. Et forte hoc senserunt quidam pytagorici naturali inclinatione ducti non ausi tamen sub talibus verbis exprimere sed in numeris sub methaphora loquebantur.*