Dynamic Models of the Term Structure

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In the past 25 years, tremendous progress has been made in modeling the dynamics of the term structure of interest rates, which play an instrumental role in determining prices and hedging portfolios of fixed-income derivative securities. This article reviews the theoretical development of the dynamic models of the default-free term structure and their applications in pricing interest rate options. Classic models, sometimes termed equilibrium models, and their multifactor extensions are outlined. These models provide clear economic intuitions connecting the term structure with economic fundamentals. They also lay a foundation for the framework of the arbitrage models that price interest rate derivatives on the basis of the market prices of bonds. This framework has been expanded and enriched by recent advances in directly modeling observable market rates through the market models and in incorporating an internally consistent correlation structure through the "infinite-dimensional" models.

A term structure of interest rates is a set of yields on discount bonds (i.e., zero-coupon bonds) with a sequence of maturing dates. Most term structures are calculated from the observed prices of government securities, such as Treasury bonds and bills in the United States, which are generally regarded as default free in developed countries. The shape of the term structure varies over time. Most of the time, the term structure is upward sloping, meaning yields on long-term bonds are higher than those on short-term bonds. The term structure can also be downward sloping, however, as it was in the United States in 1973 and the early 1980s, when short-term yields were above long-term yields. For much of 2000, the term structure was hump shaped; yields on intermediate-term notes (2–5 years) were higher than yields on both long-term (10–30 years) bonds and short-term (up to 1 year) bills.

The dynamics of the term structure of interest rates play an instrumental role in determining prices and hedging portfolios of many fixed-income derivative products. This article provides a review of the significant progress in modeling the dynamics of the default-free term structure of interest rates since the late 1970s. Although a number of excellent volumes are available on this subject,[1] my aim is to offer a coherent and up-to-date account in a multilayered structure of major developments and recent advances. I first provide a nontechnical overview of various dynamic models of the term structure. Then, after a brief summary of the technical ideas and notations commonly used in these models, I go into a detailed discussion of the models and their applications in pricing fixed-income securities.[2]

Overview

Government bonds, such as U.S. Treasury securities, are financial instruments that provide fixed and certain cash flows (coupon and principal payments) on a sequence of prespecified dates. The zero-coupon yields corresponding to various maturities of these bonds can be deduced by a method called "bootstrapping" from the market prices of the most frequently traded coupon-bearing bonds, sometimes referred to as "benchmark" issues. This set of interest rates constitutes a term structure, or a yield curve, which can have different shapes over time.[3] Other Treasury bonds or certain cash flows may be priced relative to the benchmarks; they are then said to be "priced off the yield curve." If we can obtain the current term structure from the market directly, why then do we need term-structure models?

Unlike government bonds themselves, most interest rate *derivative* securities, such as a call option on a 20-year T-bond, have payoffs that are neither fixed nor certain. These payoffs depend on either the future prices of the underlying government bonds or future levels of interest rates, all of which are unknown at the time of valuation. Valuation of interest rate derivatives thus requires

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specific assumptions about the evolution of future interest rates, whereas such assumptions are not necessary for the relative valuation of a "straight bond" in the current market.[4] For valuing derivatives, we need to model the dynamics of interest rate evolution.

The development of term-structure models has been marked by several milestones. Earlier researchers recognized the importance of the stochastic nature of interest rates and modeled the evolution of the short rate as a random walk. Vasicek (1977) introduced a general no-arbitrage framework for bonds and examined a particular model of term-structure evolution in which the short rate is mean reverting. Cox, Ingersoll, and Ross (CIR, 1985) showed how to cast the term-structure theory in a well-defined economic environment and constructed a model for positive interest rates. The CIR model retains the mean reversion in the short-rate dynamics but—unlike the Vasicek model, which assumes a constant variance—allows the variance of short-rate changes to be proportional to the level of the short rate. Both of these models prescribe a specific structure for the dynamics of interest rates and strive for a description of systematic variations of the term structure based on economic fundamentals.

The term-structure literature often classifies the Vasicek and CIR models as "equilibrium models" because they explicitly specify the market prices of risk and can be supported by an economic equilibrium. Numerous authors have extended the models into a setting in which the dynamics of the term structure are driven by multiple factors. In addition to the instantaneous short rate, authors have used such factors as a random long-run mean that the short rate is reverting to, stochastic volatility of the short rate, and interest rates of various maturities. Like the Vasicek and CIR models, these models can, in principle, be estimated from historical data on interest rates and bond prices and then be used to price both government bonds and bond options.[5] The bond prices produced by these models are likely to be different, however, from the corresponding market prices at any given time. And although this characteristic may help bond investors spot possible mispricing in bonds, based on the model assumptions about the behavior of the economic fundamentals, this feature is not desirable for pricing interest rate derivatives.

In practice, pricing interest rate derivatives requires matching the model bond prices to the current term structure. The reason is that trading derivatives usually involves simultaneously hedging the risk exposure by using the underlying securities. So, to the extent that a hedging portfolio can be constructed, the derivative price should be based, to avoid arbitrage opportunities, on the market price of the underlying security. To this end, Heath, Jarrow, and Morton (HJM, 1992) established a framework for pricing interest rate derivatives that depends on the evolution of the entire forward-rate curve, starting from the current market curve. This approach builds on the intuition in the Ho–Lee model (1986) and uses the no-arbitrage condition to pin down the relationship between the drift and the diffusion of the forward rate. The framework has a general structure and embeds many popular models as special cases. Much effort has been expended to efficiently implement derivative-pricing models in this framework. This class of models is typically termed "arbitrage models" (or "arbitrage-free models") and will be discussed in detail later.

The implementation of the HJM framework for pricing interest rate derivatives has encountered several difficulties. One of them is that the instantaneous forward rate term structure is not directly observable; thus, the HJM models are cumbersome to apply. Moreover, continuous compounding of the instantaneous forward rate rules out the popular specification of a lognormal process in this framework. The effort to mitigate this problem has led to the market models that study the observable interest rates of finite maturities, such as London Interbank Offered Rates (LIBORs) and swap rates, directly within the HJM framework (Brace, Gatarek, and Musiela 1997; Jamshidian 1997; Miltersen, Sandmann, and Sondermann 1997). On another front, the "random field" or "stochastic string" models, developed initially by Kennedy (1994) and extended and characterized by Goldstein (2000), Santa-Clara and Sornette (2001), and Filipović (2000), describe the dynamics of the forward curve through infinite-dimensional shocks to it; specifically, each point on the forward curve is driven by its own shock. With a carefully defined correlation structure between these shocks, these "infinite-dimensional" models allow a flexible and consistent description of the evolution of forward rates that matches with the market prices at all times.

The equilibrium models and the arbitrage models have similar structures.[6] The distinction comes from the different input that is used to calibrate the model parameters. The equilibrium models explicitly specify the market prices of risk; the model parameters, assumed to be time invariant, are estimated statistically from historical data. These models are often used by economists to understand the relationship between the shape of the term structure and its forecast for future economic conditions. Traders, however, would rather use arbitrage models because these models are calibrated to match the model price of the underlying security with its market price. So, traders have to make only one bet on the derivative price—a bet based on the market price of the underlying security. I maintain this classification of models for convenience of exposition in this article.

Before the detailed review of the classical models and recent advances, the following section homes in on some important, and somewhat technical, terminology and the notations commonly used in the development of term-structure models.[7] I also introduce the no-arbitrage condition and the risk-neutral valuation methodology that are important principles in derivatives valuation. Then, I examine the equilibrium models to explain the basic structures of the models and follow that description by examining their transformation into the arbitrage models used in the industry. Throughout this article, models are presented in their continuous-time formulation for expositional clarity with minimal technical formality.[8]

Preliminaries

This section lays out a set of notions about interest rates, yields, and term structures and presents the various representations of the term structure that are conventional in term-structure modeling. The section also provides an informal introduction to the no-arbitrage condition and the risk-neutral valuation methodology that are essential in pricing fixed-income derivatives.

Notions of Interest Rates, Yields, and Term Structures. A bond that entitles its holders to a single certain cash flow of F on a preset date in the future is a *pure discount bond*. It is also a *zero-coupon bond*. The amount F is referred to as either the *principal*, the *face value*, or the *notional amount*.

Let $P(t,T)$ be the price at time t of a discount bond maturing at time T with the face value F = \$1. Then, the relationship between the price and its *continuously compounded* yield, $R(t,T)$, is

$$P(t,T) = e^{-R(t,T)(T-t)} \tag{1a}$$

or

$$R(t,T) = -\frac{\ln P(t,T)}{T-t}. \tag{1b}$$

$R(t,T)$ is also termed the *spot rate* for maturity T. In particular, the spot rate of instantaneous maturity, also known as the *short rate*, is simply the limit of $R(t,T)$ when T collapses to t—that is,

$$r(t) = \lim_{T\to t} R(t,T). \tag{2}$$

The time-t continuously compounded forward rate, $f(t,T,\tau)$, covering the future period $(T, T+\tau)$, where τ denotes a length of time, is defined by

$$\frac{P(t,T+\tau)}{P(t,T)} = \exp[-f(t,T,\tau)\tau], \tag{3a}$$

which leads to

$$f(t,T,\tau) = -\frac{\ln P(t,T+\tau) - \ln P(t,T)}{\tau}. \tag{3b}$$

As τ approaches zero, the instantaneous forward rate, $f(t,T)$, becomes

$$f(t,T) = -\frac{\partial \ln P(t,T)}{\partial T}. \tag{4}$$

Equation 4 can also be expressed in an integral form:

$$P(t,T) = e^{-\int_t^T f(t,s)ds}. \tag{5}$$

From Equation 5 and Equation 1, one deduces

$$R(t,T) = \frac{1}{T-t}\int_t^T f(t,s)ds. \tag{6}$$

Equation 6 indicates that the spot rate may be interpreted as an average of the instantaneous forward rates over the remaining time to maturity of the bond.

With these definitions, the term structure of interest rates is normally represented by a set of yields, $R(t,T)$, of default-free discount bonds of various maturities. It may also be represented by a set of prices, $P(t,T)$, of zero-coupon bonds or a set of instantaneous forward rates, $f(t,T)$. Equations 4–6 demonstrate the relationships among these representations.

Although short-maturity government bills are discount bonds, most other government securities are coupon-bearing bonds with recurring coupon payments on a sequence of prespecified dates until maturity. For default-free government securities, however, these coupon bonds can be thought of as a portfolio of zero-coupon bonds of various maturities and face values.[9] Therefore, given a full set of discount bond prices, the prices of these coupon bonds are fully determined by arbitrage. This property permits a focus on the prices of discount bonds in modeling the term structure of interest rates.

So far, we have defined the term structure of interest rates in terms of default-free yields on zero-coupon bonds. The models of the term structure, however, are often used to price interest rate derivatives such as caps, floors, and swaptions, which are primarily based on LIBORs. LIBORs are not default-free rates, although they come from financial institutions with high credit ratings. In principle, one would have to consider additional risk premiums associated with these rates. Duffie and Singleton (1999) provided a general framework for modeling defaultable bonds that is similar to the framework used for modeling the term structure of default-free rates. Because of this similarity in the modeling frameworks, the models I discuss may also be used to describe the term structure of LIBORs or swap rates.[10]

No-Arbitrage Condition and Risk-Neutral Valuation. The no-arbitrage condition and risk-neutral valuation are concepts of fundamental importance in the analysis of contingent claims. *Contingent claims* are securities whose payoffs on one or several future dates depend on realized states of the world, which are often characterized by the prevailing prices of the underlying securities (or the previous paths of these prices). Interest rate derivatives are such contingent claims because their future payoffs are tied to the levels of interest rates. Their valuation depends critically on the dynamics of the term structure and is governed by the no-arbitrage condition.

The *no-arbitrage condition* states that a strategy that provides a positive future payoff in at least one state of the world and no negative payoffs in any states of the world must have a cost greater than zero today. This condition implies that a contingent claim whose payoffs can be replicated by a portfolio of securities should have a price equal to the value of the replicating portfolio. Otherwise, an arbitrage strategy exists to exploit the mispricing.[11] This pricing-by-replication principle leads to the famed Black–Scholes (1973) formula for pricing European stock options[12] and is the foundation for the pricing frameworks for interest rate derivatives.

The feasibility of replicating payoffs from a derivative product by using the underlying security and cash allows a complete hedge against the risk exposure of the derivative product, which enables application of a powerful methodology—*risk-neutral valuation*. This method of risk-neutral valuation involves calculating the expectation of the discounted payoffs from a security with a particular probability measure as if all investors were risk neutral (i.e., indifferent to risk). It thus implies that the expected return on the security in this fictitious world is simply the risk-free rate, $r(t)$.

The existence of this fictitious world with the risk-neutral probability measure is guaranteed by the no-arbitrage condition.[13] Under the risk-neutral measure, Q, the current value, V_0, of a security that pays off V_T at time T is the expectation of the discounted future payoff:

$$V_0 = E_0^Q\left[e^{-\int_0^T r(s)ds} V_T\right], \tag{7}$$

where $r(s)$ is the future short rate at time $s < T$. Note that under this measure, the discount factor (also called "*numeraire*") is an instantaneously compounded money market account, $\exp\left[\int_0^T r(s)ds\right]$, and the discounted security value is a martingale, which means that its expected future value is simply its current value.

The risk-neutral valuation method exploits the simplicity of its expectation expression, which is free of preference parameters. The connection between the risk-neutral measure and the real-world measure is established through the market price of risk, namely, the required compensation in expected excess return over the risk-free rate for bearing a unit of risk as measured by the volatility of returns. Note that in the models discussed in this article, the equilibrium models start with a specification of the market price of risk in order to arrive at the valuation of securities whereas the arbitrage models begin directly under the risk-neutral measure to obtain the valuation formulas.

In addition to the risk-neutral measure, other measures may be convenient to use in valuing interest rate derivatives. An example is the forward measure, under which the discounting *numeraire* is the price of a discount bond maturing at time T, $P(t,T)$, and the discounted value of a tradable security is a martingale. Moreover, the forward rate maturing at time T is also a martingale under this measure (i.e., its drift is zero). In this article, the risk-neutral measure or the forward measure is used in different models of the term structure.

Equilibrium Models

Equilibrium models start with specific assumptions about the dynamic processes of state variables that describe the state of the economy; the models then portray the behavior of the term structure of interest rates in such an economic environment. An important aspect is that the market prices of risk are specified explicitly in these models. Although the models are rarely used directly in industry practice, they offer economic insights into the dynamic evolution of the term structure and often form the foundation for the arbitrage models that have found widespread applications in pricing interest rate derivatives. In this section, the discussion of one-factor models is followed by an examination of their extensions to a multifactor setting, including a general class of affine models.[14]

One-Factor Models. One-factor models of the term structure of interest rates are popular because of their structural simplicity. Empirical evidence has shown that almost 90 percent of the variation in the changes of the yield curve is attributable to the variation in the first factor, which is considered to correspond to the level of the interest rate.[15] Because the first factor relates to the interest rate level, any point on the yield curve may be used as a proxy for it. For most one-factor models, the factor is generally taken to be the instantaneous short rate, $r(t)$.

The dynamics of the short rate are described by the following stochastic differential equation:

$$dr(t) = \mu(r)dt + \sigma(r)dW(t), \tag{8}$$

which means that the change in the short rate can be separated into a drift over the time period $(t, t + dt)$—namely, $\mu(r)dt$—and a random shock represented by an increment of a Brownian motion, $dW(t)$, with an instantaneous volatility of $\sigma(r)$. Note that the interest rate itself is not a traded asset but a discount bond is. The price of a discount bond, $P(t,T)$, is a function of the short rate $r(t)$. The return on the bond can be expressed as

$$\frac{dP(t,T)}{P(t,T)} = \mu_P(t,T)dt + \sigma_P(t,T)dW(t), \tag{9}$$

where the expected return on the bond, $\mu_P(t,T)$, is directly related to the drift, $\mu(r)$, and volatility, $\sigma(r)$, of the short rate and the volatility of the bond return is related to $\sigma(r)$.[16]

The no-arbitrage condition applied to the set of discount bond prices requires that

$$\frac{\mu_P(t,T) - r(t)}{\sigma_P(t,T)} = \lambda(r), \tag{10}$$

where $\lambda(r)$ is the market price of risk. The market price of risk is the required compensation in the form of expected excess return over the risk-free rate for bearing a unit of risk as measured by the volatility of return. It should be the same for all bonds in the economy (i.e., it is independent of maturity date T). The specification of $\lambda(r)$ differs among models because it depends on additional assumptions about investor preferences and production technologies or endowment processes in an economy.

Once the market price of risk has been determined, the process for the short rate under risk-neutral probability measure Q can be expressed as

$$dr = [\mu(r) - \lambda(r)\sigma(r)]dt + \sigma(r)dW^Q(t) \tag{11}$$

and the risk-neutral process for the bond price becomes

$$\frac{dP(t,T)}{P(t,T)} = r(t)dt + \sigma_P(t,T)dW^Q(t). \tag{12}$$

Notice that under the risk-neutral measure, all traded securities have their instantaneous expected returns equal to the risk-free rate.[17]

Various one-factor models are constructed by specifying the drift, $\mu(r)$, and volatility, $\sigma(r)$, of the short rate. The market price of risk, $\lambda(r)$, is also determined in a model. Then, prices of bond and bond options can be obtained in the risk-neutral valuation framework. For instance, because the payoff for holding a discount bond with a face value of \$1 maturing at time T is receiving a certain dollar at maturity, the price of the bond is simply

$$P(t,T) = E_t^Q\left[e^{-\int_t^T r(s)ds} 1\right], \tag{13}$$

with the expectation taken under the risk-neutral probability measure, Q. In general, the price for an interest rate derivative security promising a payoff flow $g(r, \tau, T)$ with $t < \tau < T$ is given by

$$V(t,r) = E_t^Q\left[\int_t^T e^{-\int_t^\tau r(s)ds} g\{r(\tau),\tau,T\}d\tau\right]. \tag{14}$$

■ *The Vasicek model.* Vasicek was the first to study the term structure with mean-reverting short-rate dynamics. Although the model bearing his name was initially constructed as a special example to illustrate the arbitrage-free pricing framework, it has since gained lasting influence and popularity.

In the Vasicek model, the short rate follows an Ornstein–Uhlenbeck (O–U) process as follows:

$$dr(t) = \kappa[\bar{r} - r(t)]dt + \sigma dW(t), \tag{15}$$

where κ measures the speed of mean reversion, $\bar{r}$ is the long-run mean to which the short rate is reverting, and σ is the instantaneous volatility of the short rate; all are assumed constant. Because the conditional distribution of $r(t)$ following an O–U process is Gaussian, the Vasicek model is also referred to as a Gaussian model.

In this model, the market price of risk is a constant [i.e., $\lambda(r) = \lambda_0$]. Hence, the risk-neutral process for the short rate is

$$dr(t) = \kappa[\bar{r}' - r(t)]dt + \sigma dW^Q(t), \tag{16}$$

with $\bar{r}' = \bar{r} - (\lambda_0\sigma)/\kappa$, which indicates that the process for the short rate under the risk-neutral measure is similar to the process in the real measure except for a shift in the long-run mean. The price of a discount bond is shown to be

$$P(t,T) = \exp[A(\tau) - B(\tau)r(t)], \tag{17}$$

where $\tau = T - t$. Notice that the discount bond price is exponentially linear in short rate $r(t)$. Given the relationship between discount bond price and continuously compounded yield in Equation 1, this implies that spot rates of all maturities, $R(t,T)$, are linear in $r(t)$. The deterministic functions $A(\tau)$ and $B(\tau)$ relate spot rates of varying maturities to the short rate.[18] This simple expression also makes it easy to obtain formulas for prices of European options on zero-coupon bonds in this model.[19]

The Vasicek model produces term-structure shapes that can be either upward sloping, downward sloping, or humped. The relative simplicity of the model structure and the incorporation of mean reversion in the interest rate dynamics have made the model influential ever since it was first published. The model is subject to generating negative interest rates, however, because of the Gaussian distribution. This characteristic is not necessarily a problem for real interest rates, but it is troublesome for modeling nominal rates and pricing interest rate derivatives (see, e.g., Rogers 1996).

■ *The Cox–Ingersoll–Ross model.* In a general equilibrium framework, Cox, Ingersoll, and Ross constructed a term-structure model that uses a square root process for the short rate:

$$dr(t) = \kappa[\bar{r} - r(t)]dt + \sigma\sqrt{r(t)}dW(t). \tag{18}$$

The change in the short rate has a mean-reverting drift and a variance that is proportional to the level of the interest rate. This process has a reflecting boundary at $r(t) = 0$ if $2\kappa\bar{r} \geq \sigma^2$. Hence, it can preclude negative short rates.

The market price of risk in this model now depends on the short rate because $\lambda(r) = \lambda_0\sqrt{r}/\sigma$. Under risk-neutral measure Q, the short-rate process becomes

$$dr(t) = \kappa'[\bar{r}' - r(t)]dt + \sigma\sqrt{r(t)}dW^Q(t), \tag{19}$$

which now has a mean-reverting speed of $\kappa' = \kappa + \lambda_0$ and a long-run mean of $\bar{r}' = \kappa\bar{r}/(\kappa + \lambda_0)$. The price of a discount bond with time to maturity of $\tau = T - t$ has the familiar form of Equation 17.[20] As in the Vasicek model, the bond price in the CIR model is also exponentially linear in the short rate. In fact, this characteristic is common to a general class of affine models, as will be discussed later. The CIR model can also accommodate a variety of shapes for the yield curve. Cox, Ingersoll, and Ross provided a closed-form formula for pricing European options on discount bonds.

Single-factor models, such as the Vasicek and CIR models, describe the evolution of the term structure of interest rates in a simple way. They assume, however, that the dynamics of all bonds are driven by the same source of random shocks and, therefore, that spot rates are locally perfectly correlated with each other. This assumption is counterfactual; empirical studies have shown that (1) correlations between various yields are different from unity and (2) yields are highly correlated if they have similar times to maturity but their correlations are significantly reduced if they are in different segments of the yield curve. Furthermore, empirical evidence suggests a more complex short-rate volatility structure than either of these models can accommodate (see, e.g., Chapman and Pearson 2001). This point is particularly important because the value of an interest rate derivative critically depends on the specification of the volatility structure. These problems highlight the necessity for multifactor models of the term structure.

Multifactor Models. Multifactor models postulate that the evolution of the term structure of interest rates is driven by the dynamics of several factors and, therefore, the yields are functions of these factors. These factors can be represented by macroeconomic shocks or be related to the level, slope, and curvature of the yield curve itself. Empirical research (Litterman and Scheinkman 1991) has bolstered the intuition behind the multifactor models. In the last two decades, various forms of multifactor models have been proposed and studied. Here, a review of some representative two-factor models is followed by discussion of the general characteristics of affine models, in which the yields are linear functions of the factors.

■ *The Brennan–Schwartz model.* Brennan and Schwartz (1979) developed a two-factor model based on the dynamics of two yields on the curve. The two factors are represented by the short rate, $r(t)$, and the console yield, $l(t)$.[21] They are governed by the following dynamics:

$$dr(t) = \beta_1(r,l,t)dt + \eta_1(r,l,t)dW_1(t) \tag{20a}$$

and

$$dl(t) = \beta_2(r,l,t)dt + \eta_2(r,l,t)dW_2(t), \tag{20b}$$

where $\beta_1(r,l,t)$ and $\beta_2(r,l,t)$ are the drift terms and $\eta_1(r,l,t)$ and $\eta_2(r,l,t)$ are the volatility terms of, respectively, the short rate and the console yield and W_1 and W_2 are two correlated Brownian motions. Note that the difference between the long rate and the short rate proxies for the slope measure of the term structure. So, this model should account for both level and slope effects of the term structure. A market price of risk is associated with each risk factor. Once the functional forms of the market prices of risk and the drift and volatility terms in the processes for short and long rates are specified, bond prices can be determined.

The functional forms in this model can be specified in various ways. Generally, there are no closed-form solutions for bond prices unless the functional forms are specified to be affine in two factors (to be discussed). For a model involving the console yield, however, Dybvig, Ingersoll, and Ross (1996) prescribed a stringent test for the absence of arbitrage, namely, that the long forward and zero-coupon rates can never fall. Whether any specification of the Brennan–Schwartz model will satisfy this requirement is not clear. Hogan (1993) showed that some specifications of the Brennan–Schwartz model may lead to an infinite long yield with positive probability in finite time that is inconsistent with the no-arbitrage condition.

If we move away from the console yield and, instead, model any two yields of finite maturities in the same fashion as the short rate and console yield are described in the Brennan–Schwartz model, then we can generate a whole class of term-structure models of practical significance. In particular, if we specify the functional forms to be those in the Vasicek and CIR models, we can obtain the prices of bonds and bond options in closed form, as demonstrated in Langetieg (1980) and Chen and Scott (1992).

■ *The Fong–Vasicek model.* Empirical studies have revealed that the volatility of the changes in the short rate is time varying and stochastic. To explicitly model the stochastic changes in the interest rate volatility and their effect on bond prices and option values, Fong and Vasicek (1991) proposed a two-factor extension of the Vasicek model in which the O–U process for the short rate is modified to include a stochastic variance that follows a square-root process:

$$dr(t) = \kappa_1[\bar{r} - r(t)]dt + \sqrt{V(t)}dW_1(t) \quad (21a)$$

and

$$dV(t) = \kappa_2[\bar{V} - V(t)]dt + \eta\sqrt{V(t)}dW_2(t), \quad (21b)$$

where Brownian motions $W_1(t)$ and $W_2(t)$ are correlated. The market price of risk for each factor is specified as $\lambda_i(t) = \lambda_i\sqrt{V(t)}$ for $i = 1, 2$. Variables κ_1 and κ_2 describe the speed at which the short rate, $r(t)$, and its variance, $V(t)$, revert to their long-run means, respectively, $\bar{r}$ and $\bar{V}$. The price of a discount bond is exponentially linear in both $r(t)$ and $V(t)$, although numerical methods need to be used to obtain the exact prices for bonds and bond options.

■ *The Longstaff–Schwartz model.* Another two-factor model that describes the dynamics of the short rate and its variance was developed by Longstaff and Schwartz (1992) within the CIR general equilibrium framework. In the Longstaff–Schwartz model, two state variables, X and Y, represent the state of the economy; each follows a square-root process as in the CIR model:[22]

$$dX = (a - bX) + c\sqrt{X}dW_1 \quad (22a)$$

and

$$dY = (d - eY) + f\sqrt{Y}dW_2, \quad (22b)$$

where a, b, c, d, e, and f are specific parameters in the model and dW_1 and dW_2 are independent Brownian motions. Given the structure of the model, the short rate, $r(t)$, is linear in the state variables and so is its instantaneous variance, $V(t)$. From these relationships, one can derive processes that depict the dynamics of the short rate and its variance in the same spirit of the Fong–Vasicek model.

The processes for the two factors, short rate and its variance, are complicated in the Longstaff–Schwartz model, but all the drift and variance terms are linear in these two factors. Again, this feature is characteristic of affine models. The price of a zero-coupon bond is shown to be exponentially linear in r and V; that is,

$$P(t,T) \propto \exp[C(\tau)r + D(\tau)V], \quad (23)$$

where $C(\tau)$ and $D(\tau)$ relate the bond price to the state variables, r and V, and can be obtained analytically. Furthermore, one can derive the prices of European-style bond options in closed forms, which permits easy calibration with the aid of computers.

■ *Multifactor affine models.* The preceding examples illustrate a variety of two-factor models. In practice, the number of factors needed in a model to effectively price and hedge interest rate derivatives may be more than two. Litterman and Scheinkman found three factors (corresponding to the level, slope, and curvature of the yield curve) that drove the term structure in the Treasury market in the 1980s, and Longstaff, Santa-Clara, and Schwartz (2000) proposed four significant factors at play in the LIBOR market in recent years.

Despite the increased complexity, the addition of more factors does not much change the general structure of multifactor models. This subsection focuses on affine models in which yields are linear functions of the factors. This class of models has received a lot of attention in recent years because it includes many popular models and offers superior tractability.[23]

Suppose that in a multifactor model, there are N factors, $X_1, X_2, \ldots, X_N$. Let $\mathbf{X}$ denote the vector $(X_1, X_2, \ldots, X_N)'$ that evolves over time following a multidimensional diffusion process:

$$d\mathbf{X}(t) = \mu[\mathbf{X}(t)]dt + \sigma[\mathbf{X}(t)]d\mathbf{W}(t), \quad (24)$$

where $\mu[\mathbf{X}(t)]$ is an N-dimensional vector, $\sigma[\mathbf{X}(t)]$ is an $N \times N$ matrix, and $\mathbf{W}(t)$ is a vector of N independent Brownian motions.

Many of the factor models in the literature fall into a general class of affine models (see Duffie and Kan 1996). In an affine model, the instantaneous short rate is a linear combination of the factors:

$$\begin{aligned} r(t) &= \delta_0 + \sum_{i=1}^{N} \delta_i \mathbf{X}_i(t) \\ &= \delta_0 + \delta' \mathbf{X}(t), \end{aligned} \tag{25}$$

where the δ's are constant coefficients. The drift and variance–covariance matrix for the factors are also affine functions (i.e., linear functions up to a deterministic term) of the factors, the $\mathbf{X}(t)$'s. Therefore, one can generally specify $\mu[\mathbf{X}(t)]$ and $\sigma[\mathbf{X}(t)]$ as

$$\mu[\mathbf{X}(t)] = \tilde{\mathbf{K}}[\theta - \mathbf{X}(t)] \tag{26a}$$

and

$$\sigma[\mathbf{X}(t)] = \Sigma\sqrt{\mathbf{S}(t)}, \tag{26b}$$

where θ is a constant vector representing the long-run mean that $\mathbf{X}(t)$ is reverting to, $\tilde{\mathbf{K}}$ and Σ are $N \times N$ matrixes that may be nondiagonal and asymmetrical, and $\mathbf{S}(t)$ is a diagonal matrix with its ith diagonal element given by

$$[\mathbf{S}(t)]_{ii} = \alpha_i + \beta_i'\mathbf{X}(t). \tag{27}$$

Hence, both the drift, $\mu[\mathbf{X}(t)]$, and the conditional variance–covariance matrix, $\sigma[\mathbf{X}(t)]'\sigma[\mathbf{X}(t)]$, are affine in $\mathbf{X}(t)$.

The elements of $\mathbf{X}(t)$ in this setup may be macroeconomic variables, as in the Longstaff–Schwartz model where the short rate and its variance are linear in these state variables. Indeed, Cox, Ingersoll, and Ross proposed a multifactor extension of their model in a similar form as presented here, which was further studied by Chacko and Das (1999), Chen and Scott, Duffie and Singleton, and Pearson and Sun (1994). Similar extensions can also be applied to the Gaussian model of Vasicek (see Langetieg). Duffie and Kan showed that, the zero-coupon bond prices in these models are given by

$$P(t,T) = \exp[A(\tau) - \mathbf{B}(\tau)'\mathbf{X}(t)], \tag{28}$$

with $\tau = T - t$, where $A(\tau)$ and $\mathbf{B}(\tau)$ satisfy a set of ordinary differential equations (ODEs) with proper initial conditions. Although closed-form solutions to this set of ODEs are attainable only in special cases, solving the ODEs numerically is generally easy.

Note that the formulation presented here can be simply framed in the risk-neutral measure. Then, if the market price of risk is assumed to be

$$\Lambda[\mathbf{X}(t)] = \lambda\sqrt{\mathbf{S}(t)}, \tag{29}$$

the processes for the state variables in the true probability measure, where empirical measurements are made, will also follow affine diffusions. This step will facilitate empirical estimation of these models.[24] Because the affine processes are specified for the generic state variables and the spot yields are linear in these state variables (hence, become proxies for these state variables), this type of model was labeled an "AY model" (for "affined yields") by Dai and Singleton (2000).

Another set of multifactor models focuses on the short rate with a stochastic long-run mean, or "central tendency," and/or a stochastic volatility. This type of model generally takes the following representation:

$$dr(t) = \kappa[\theta(t) - r(t)]dt + \sqrt{\nu(t)}r^{\gamma}(t)dW_r, \tag{30a}$$

$$d\theta(t) = \nu[\bar{\theta} - \theta(t)]dt + \sqrt{\alpha + \beta^2\theta(t)}dW_{\theta}, \tag{30b}$$

and

$$d\nu(t) = \mu[\bar{\nu} - \nu(t)]dt + \sqrt{\eta^2\nu(t)}dW_{\nu}, \tag{30c}$$

where $\theta(t)$ is the long-run mean; $\nu(t)$ relates to the short-rate volatility; α, β, μ, ν, and η are constant coefficients; γ generally takes a value of 0 or 1/2; and W's are Brownian motions that may be correlated. This type of model has been studied in various forms (and with different γ's) by Chen (1996), Balduzzi, Das, Foresi, and Sundaram (1996), and Fong and Vasicek. Because in these models the processes are specified for various aspects of the short rate, the short rate is affine (linear) in these variables. Therefore, this type of model was labeled an "Ar model" by Dai and Singleton. Dai and Singleton found correspondence between certain forms of AY models and Ar models. They used this identification to empirically test specifications of affine term-structure models.

In addition, the affine models may accommodate jump components in the processes for the state variables. Earlier models incorporating jump-diffusion processes include Ahn and Thompson (1988) and Das and Foresi (1996). Duffie and Kan formally specified the restrictions on the jump-diffusion processes in order to maintain the exponentially affine structure for bond prices. Chacko and Das explicitly calculated prices of interest rate derivatives within this framework, and Duffie, Pan, and Singleton (2000) provided a general treatment of a class of transforms in the setting of affine jump-diffusion processes that is readily applicable to valuing fixed-income securities. Econometric modeling of jumps in the framework of affine models can be found in Das (forthcoming) and Piazzesi (2001).

Arbitrage Models

The factor models discussed so far may be derived from some equilibrium framework, which would preclude the existence of arbitrage in the specified economy. These models are generally estimated to explain the observed historical patterns in the dynamics of the term structure, which may, in turn, help analysts understand the dynamics of the economy. This approach is not practical, however, for pricing interest rate derivatives because empirically fitted models using historical data will not guarantee that the model term structure matches the current term structure obtained from market prices. If the term structures differ, as they almost always do, a derivatives trader who uses the model instead of the prevailing term structure cannot adequately hedge derivative positions with underlying securities. For this reason, significant effort has been expended to make a factor model match the current yield curve before it is used to price options.

Matching the Current Yield Curve. One way to match the current term structure is to make the coefficients in a factor model vary deterministically with time. Cox, Ingersoll, and Ross first discussed this possibility in the context of the CIR model, and Dybvig (1997) further investigated pricing of interest rate derivatives in this framework. This type of model takes the market prices of bonds (hence, the current term structure) as given and prices interest rate derivatives accordingly. Therefore, the models will not spot mispricing in the underlying bonds themselves but are useful for pricing derivatives in the same spirit as the Black–Scholes framework for stock options. This section discusses three of the most popular models. They are popular because of the simple intuition behind them and the easy calibration they permit. Although multifactor implementation is often preferred and used, the one-factor versions of these models are presented here for clarity.

■ *The Ho–Lee model.* Starting from the premise that the short rate follows a random walk, Ho and Lee recognized that allowing the drift of the short-rate process to be time varying would accommodate an essentially arbitrary specification of the initial term structure. That is, if one assumes the short-rate process to be

$$dr = \theta(t)dt + \sigma dW, \tag{31}$$

where the instantaneous standard deviation of the short rate, σ, is constant, then matching the initial term structure to the market yield curve will help pin down the drift, $\theta(t)$.

In this continuous-time version of the Ho–Lee model, the price of a zero-coupon bond is

$$P(t,T) = e^{A(t,T)-r(t)(T-t)}, \tag{32}$$

where $A(t,T)$ is a deterministic integral of $\theta(t)$. Then, matching the current yield curve dictates that

$$\theta(t) = \frac{\partial f(0,t)}{\partial t} + \sigma^2 t, \tag{33}$$

where $f(0,t)$ is the instantaneous forward rate for time t seen at time zero.

Within this model, prices of European options on discount bonds can be easily obtained in closed form. For example, the price at time zero of a call option that expires at time τ on a discount bond maturing at time T is

$$FP(0,T)N(d) - KP(0,\tau)N(d - \sigma_P), \tag{34}$$

where F is the principal of the bond, K is the strike price of the option, and parameters d and σ_P are given as follows:

$$d = \frac{1}{\sigma_P}\ln\left[\frac{FP(0,T)}{KP(0,\tau)}\right] + \frac{\sigma_P}{2} \tag{35a}$$

and

$$\sigma_P = \sigma(T-\tau)\sqrt{\tau}. \tag{35b}$$

Note that Equation 34 is similar to the Black–Scholes formula for stock options. Here, $P(0,\tau)$ and $P(0,T)$ are market prices of discount bonds maturing at, respectively, times t and T. American-style options can be evaluated through a binomial tree implementation.[25]

■ *The Hull–White model.* Although the Ho–Lee model provides an exact fit to the current yield curve and is straightforward to implement, it does not explicitly account for mean reversion in the short rate. In addition, the volatility structure implied by the model is flat for all rates. To counter these problems, Hull and White (1990) extended the Vasicek model to match the initial term structure. One version of their extended Vasicek model gives the short-rate dynamics as

$$dr = [\theta(t) - \kappa r]dt + \sigma dW. \tag{36}$$

Equation 36 includes the Ho–Lee model as a special case when $\kappa = 0$. Bond prices at time t are then given by

$$P(t,T) = e^{A(t,T)-B(t,T)r(t)}, \tag{37}$$

where $A(t,T)$ is directly related to the time-varying drift function, $\theta(t)$. Matching the current yield curve entails calibrating $\theta(t)$ to be

$$\theta(t) = \frac{\partial f(0,t)}{\partial t} + \kappa f(0,t) + \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa t}). \tag{38}$$

The volatility structure in the Hull–White model is determined by both κ and σ and has richer variations than that in the Ho–Lee model. Specifically, the volatility at time t of $P(t,T)$ is

$$\frac{\sigma}{\kappa}\left[1 - e^{-\kappa(T-t)}\right]. \tag{39a}$$

The instantaneous standard deviation at time t of the spot rate $R(t, T)$ is

$$\frac{\sigma}{\kappa(T-t)}\left[1 - e^{-\kappa(T-t)}\right], \tag{39b}$$

and the volatility of forward rate $f(t,T)$ is $\sigma e^{-\kappa(T-t)}$. These variations in volatility structure represent a step toward realism, but matching them with observed volatilities is often still difficult.

In the Hull–White setup, prices of European options on discount bonds can be obtained in an analytic form similar to that in the Ho–Lee model. More commonly, Hull–White-type models are implemented on a trinomial tree, which greatly facilitates the numerical valuation of American-style options.[26]

■ *The Black–Derman–Toy model.* In an effort to fit not only the current bond prices but also the current volatility structure, Black, Derman, and Toy (1990) proposed building a binomial tree that is equivalent to the following process:

$$d\ln r(t) = \left[\theta(t) + \frac{\sigma'(t)}{\sigma(t)}\ln r(t)\right]dt + \sigma(t)dW(t), \tag{40}$$

where $\sigma'(t)$ is the derivative of $\sigma(t)$ with respect to t. Here, short rate $r(t)$ has a lognormal distribution; thus, this model avoids the problem of possible negative rates that is present in both the Ho–Lee and Hull–White models. The time-varying (but deterministic) function $\theta(t)$ in the drift and volatility $\sigma(t)$ are calibrated to match the initial yield curve and volatility structure.

One issue arising in the Black–Derman–Toy model is that the mean-reversion speed for the short rate, $\sigma'(t)/\sigma(t)$, is directly tied to the volatility structure. This coupling is artificial and may impose the unwarranted constraint of simultaneously matching the current yield curve and term structure of volatilities. To rectify this problem, Black and Karasinski (1991) generalized this model to the following form:

$$d\ln r(t) = [\theta(t) - a(t)\ln r(t)]\,dt + \sigma(t)dW(t), \tag{41}$$

where $a(t)$ is an independent deterministic function of time. The implementation of this model is done through a trinomial-tree-building procedure. Bliss and Ronn (1998) provided an application of this model to examine the optimal call policies and implied volatilities of callable U.S. T-bonds.

The Heath–Jarrow–Morton Approach. Heath, Jarrow, and Morton took the instantaneous forward rates as exogenously specified to derive, using the martingale measure implied by the forward rates, contingent-claim prices. This approach is equivalent to taking the dynamics of bond prices as given and pricing other interest rate derivatives on the basis of the no-arbitrage condition, in much the same spirit as the Ho–Lee and Hull–White models, as well as that of the Black–Scholes option-pricing model. Just as the Black–Scholes model assumes that the stock price follows a geometrical Brownian motion, the HJM framework specifies the dynamics of forward rates as

$$df(t,T) = \alpha(t,T)dt + \sum_{i=1}^{N}\sigma_i(t,T)dW_i(t), \tag{42}$$

where $f(t,T)$ is the instantaneous forward rate maturing at time T measured at time t and $\alpha(t,T)$ and $\sigma_i(t,T)$ are stochastic functions satisfying necessary regularity conditions. N is the number of stochastic factors that are driving the evolution of the forward curve. Alternatively, the forward process may be expressed in an integral form as

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds + \sum_{i=1}^{N}\int_0^t \sigma_i(s,T)dW_i(s) \tag{43}$$

for $0 \le t \le T$. Here, $f(0,T)$ represents the initial (implied) market forward curve.

Unlike stocks, forward rates are not tradable assets. Rather, the initial forward curve is obtained from a set of traded bond prices. Because of the relationship between bond price and forward rate, set forth in Equation 5, one can also derive the process for bond prices. The no-arbitrage condition and the assumption of a complete market[27] then imply that (1) the market prices of risk, $\lambda_i(t)$, are uniquely determined by and contained in the market prices of bonds and (2) there is a specific relationship between the drift and the volatility of the forward-rate process. That is,

$$\alpha(t,T) = \sum_{i=1}^{N}\sigma_i(t,T)\left[\int_t^T \sigma_i(t,s)ds - \lambda_i(t)\right]. \tag{44}$$

Therefore, the drift of the forward rate is determined entirely by its volatility structure under both the physical measure and the risk-neutral measure.[28]

The specification of an HJM model depends critically on the specification of the volatility structure of forward rates. Consider a simple one-factor example ($N = 1$) in which we assume that $\sigma_1(t,T) = \sigma$ is a constant (i.e., the volatility structure for forward rates is flat). Then, Equation 44 implies that $\alpha(t,T) = -\sigma\lambda_1(t) + \sigma^2(T - t)$. Under the risk-neutral measure, the instantaneous forward rate evolves as

$$f(t,T) = f(0,T) + \sigma^2 t\left(T - \frac{t}{2}\right) + \sigma W^Q(t). \tag{45}$$

The bond price process becomes

$$\frac{dP(t,T)}{P(t,T)} = r(t)dt - \sigma(T-t)dW^Q(t), \qquad (46)$$

where $r(t) = f(t,t)$ is the short rate. Note that Equation 46 corresponds to the specification of the Ho–Lee model discussed previously.

Beyond a few examples like this one, however, in which the volatility structure of forward rates or bond returns is unrealistically simplistic, this approach does not produce closed-form formulas for interest rate derivative prices. Even numerical procedures can be complicated because, with a general form for volatility functions $\sigma_i(t,T)$, the evolution of the bond price depends on the whole history of interest rates. This path dependency makes the implementation of these models difficult. On the one hand, the tree structure used in many numerical procedures becomes nonrecombining, or "bushy." This type of bushy tree quickly explodes in size, which poses problems for evaluating long-term contracts. On the other hand, traditional Monte Carlo simulation techniques are generally inept in dealing with American-style options.[29]

To circumvent this problem, a class of Markovian HJM models has been proposed that impose more structure on the volatility functions (Ritchken and Sankarasubramanian 1995; Inui and Kijima 1998). Specifically, if one restricts the volatility functions, $\sigma_i(s,t)$, to solve

$$\frac{\partial \sigma_i(s,t)}{\partial t} = -\kappa_i(t)\sigma_i(s,t), \qquad (47a)$$

where $\kappa_i(t)$ is an arbitrary but deterministic function, and if one defines

$$\phi_i(t) = \int_0^t \sigma_i^2(s,t)ds, \qquad (47b)$$

then the processes for short rate $r(t)$ and for $\phi_i(t)$ become jointly Markovian. Thus, one can build a tree structure that is once more recombining, as desired in numerical implementations.

It is not surprising that the models discussed previously, all of them Markovian, are special cases in this class of specification. For example, if we assume that $\kappa_i(t) = \kappa_i$ and $\sigma_i(s,s) = \sigma_i$, then this class of Markovian HJM models reduces to the Hull–White model. When κ is zero, the Ho–Lee model is recovered.

Recent Advances

The development of the HJM framework for term-structure models stimulated tremendous interest and effort in finding practical ways to implement the HJM models to effectively price derivative products. These recent activities attempt to address problems associated with various aspects of the HJM models.

For instance, negative interest rates are not *a priori* excluded because, not only are Gaussian models admissible in the HJM framework, but calibrating parameters to fit the current term structure can result in losing the positivity of interest rates.[30] Recognizing this problem, Flesaker and Hughston (1996) proposed a formulation based on specifying a family of positive martingales. This elegant construction of interest rate dynamics ensures positive rates and is consistent with the HJM framework. Remarkably, various special cases of this model can be formulated for which the bond option prices, as well as the prices of caps and swaptions, can be derived analytically in the Black–Scholes-type formulas.[31]

This section discusses two recently proposed classes of models—market models, which deal directly with observable market rates, and infinite-dimensional models, which appear to model the correlation structure properly in a parsimonious way.

Market Models. The models popularly known as market models deal directly with observable market rates, such as LIBOR or swap rates of finite maturities. In these models, forward rates of finite durations are assumed to be (conditionally) lognormally distributed. With properly matched volatilities, this approach recovers the widely used Black (1976) model in an internally consistent framework.[32] This class of models has found widespread application in the industry.

On the one hand, the HJM approach exogenously specifies the dynamics for instantaneous forward rates with their current term structure and volatility structure as inputs to price interest rate options. This approach poses a problem for practical implementation because the current term structure of *instantaneous* forward rates, $f(t,T)$, and its volatility structure are inherently unobservable. The differential relationship between $f(t,T)$ and $P(t,T)$ makes a direct translation from observed bond prices to instantaneous forward rates difficult. One may use proxies, such as one-month or three-month T-bill or LIBOR yields or their forward rates, but these proxies can be misleading (Chapman, Long, and Pearson 1999). On the other hand, although the lognormal process is popular because it ensures a positive interest rate process and enables a closed-form formula for option prices, it is not suitable for instantaneous forward rates; continuous compounding will render an explosion in bond prices, violating the no-arbitrage conditions.

The class of market models developed by Brace, Gatarek, and Musiela (BGM), Jamshidian (1997), and Miltersen, Sandmann, and Sondermann (MSS) has emerged to overcome these problems. In their various representations, the models are expressed in terms of forward rates with finite durations in the LIBOR or swap market, named "market variables." These models also lend justification to the market practice of using the Black model to price caps and floors or swaptions.

To see how these models work, consider a one-factor rendition and denote the *simply compounded* forward rate for a loan between time T and time $T + \delta$ set at time t as $F(t,T,\delta)$, which is determined by two discount bond prices $P(t,T)$ and $P(t,T+\delta)$ through[33]

$$P(t,T+\delta) = \frac{P(t,T)}{1+\delta F(t,T,\delta)}. \qquad (48)$$

The market models postulate that the process for $F(t,T,\delta)$ is

$$\frac{dF(t,T,\delta)}{F(t,T,\delta)} = \mu(t,T,\delta)dt + \gamma(t,T,\delta)dW(t), \qquad (49)$$

which is initiated with the observed term structure of interest rates at $t = 0$:

$$F(0,T,\delta) = \frac{1}{\delta}\left[\frac{P(0,T)}{P(0,T+\delta)} - 1\right]. \qquad (50)$$

As long as $\delta \neq 0$, this specification will not cause the forward-rate explosion.[34]

Clearly, once the selected nonredundant forward rates are described by lognormal processes, other rates that can be replicated by these basis rates will not be described by lognormal processes because the sum of lognormal variables does not follow a lognormal distribution. Both the BGM and MSS models calibrate to the LIBOR market with $F(t,T,\delta)$ as the forward LIBOR rate, but the model of Jamshidian (1997) also calibrates to the swap market, with $F(t,T,\delta)$ being the forward swap rate. Because swap rates may be viewed (approximately) as weighted averages of forward LIBORs, if forward LIBORs are lognormally distributed, swap rates cannot be. Thus, although these different representations of the market model share a similar structure, they cannot be simultaneously compatible with each other. Therefore, one has to choose a representation by considering the context of the valuation.[35]

It can be shown that in the LIBOR market, under the forward measure with $P(t,T+\delta)$ as the *numeraire*, $F(t,T,\delta)$ is a martingale:

$$\frac{dF(t,T,\delta)}{F(t,T,\delta)} = \gamma(t,T,\delta)dW^{(T+\delta)}(t), \qquad (51)$$

where $dW^{T+\delta}(t)$ is a Brownian motion under this forward measure indexed by maturity date $T + \delta$. This immediately yields a formula for the price of a caplet with strike rate L at time t, CPL_t:

$$CPL_t = \delta P(t,T,\delta)[F(t,T,\delta)N(d_+) - L\,N(d_-)], \qquad (52)$$

where

$$d_\pm = \frac{1}{\sigma(t,T,\delta)}\left[\ln\frac{F(t,T,\delta)}{L} \pm \frac{\sigma^2(t,T,\delta)}{2}\right] \qquad (53a)$$

and

$$\sigma^2(t,T,\delta) = \int_t^T \gamma^2(s,T,\delta)ds. \qquad (53b)$$

Equation 52 is exactly the Black formula commonly used in industry practice if $\sigma(t,T,\delta)$ matches the (appropriately scaled) cap volatility.

The market models, especially their multifactor versions, have found widespread application in pricing interest rate derivative products because they make direct connections between option prices and "market" rates (such as LIBOR and swap rates) and justify the popular use of the Black formula with clear guidance for setting appropriate variance terms. These models are consistent with the general framework of the HJM approach, but the simplicity in structure of the market model results in closed-form solutions to cap / floor or swaption prices and makes it easier to price and hedge more complicated derivatives, such as barrier caps, chooser caps, and flexi caps. One caveat for users of market models is that the model is only as good as its assumption of the (finite-duration) forward-rate process, the validity of which is still an open empirical question.

Infinite-Dimensional Models. One of the most important problems in modeling the term structure and pricing interest rate derivatives is to properly account for the correlation structure between rates of different maturities. This problem directly affects hedging strategies. In the multifactor models discussed previously, the correlation structure is determined by a limited number of factors, which may not reflect the natural relationship between bonds in a similar maturity segment. This problem is addressed by a class of infinite-dimensional models, referred to sometimes as "random-field" or "stochastic-string" models, which show considerable promise in properly and parsimoniously modeling the correlation structure.

The models in the HJM framework are purported to be consistent with the initial term structure. But if the parameters are fixed at presently calibrated values, these models are not guaranteed to be consistent with future innovations to the term structure. In fact, implementation of these models is subject to continuous recalibration of the model parameters, even though the *a priori* assumptions of the models call for *deterministic* coefficients. Therefore, there seems to be inherent inconsistency between the assumptions of the models and their practical application. Part of the reason for this inconsistency may be that extant term-structure models have the same set of shocks affecting all forward rates, which constrains the correlations between bond prices of different maturities and limits the set of admissible shapes and dynamics of the yield curve. The random-field models (Kennedy 1994 and 1997 and Goldstein) or stochastic-string models (Santa-Clara and Sornette) were developed to address this problem.

In these models, each instantaneous forward rate is driven by its own shock, ordered by maturity, and each of these shocks is imperfectly correlated with shocks to other instantaneous forward rates of different maturities. The dynamics of instantaneous forward rates, similarly exogenously specified as in the HJM models, is defined as

$$df(t,T) \equiv \mu_T(t)dt + \sigma_T(t)dZ_T(t). \tag{54}$$

The shock, or innovation, $dZ_T(t)$, indexed by maturity date T,[36] generalizes a one-dimensional Brownian motion to a two-dimensional Brownian field (or string shock).[37] (Note that a model so constructed has infinite dimensions because each point on the forward curve is driven by its own shock.) The seemingly simple notational change in the stochastic disturbance makes an important difference because $dZ_T(t)$ not only denotes the differential magnitude of shocks to different points on the forward curve but also embeds a correlation structure between these shocks. For example, one could have the following correlation structure between two string shocks to two forward rates indexed by T_1 and T_2:

$$\text{corr}[dZ_{T_1}(t), dZ_{T_2}(t)] = e^{-\rho|T_1 - T_2|}. \tag{55}$$

This type of construction enables the model to match the dynamics of the forward curve at all times, which is not the case for the traditional multifactor models. In the traditional models, if the number of bonds is more than the number of factors, a calibration is necessary at each time to fit the term structure as closely as possible.

An infinite-dimensional model does not, however, have an infinite number of factors. Although a semantic dispute about the difference between a "factor" and a "source of random shocks" is possible, the preceding example demonstrates that a rich and plausible correlation structure can be generated in an infinite-dimensional model without a large number of parameters to estimate (as in a traditional multifactor model).

Under the risk-neutral measure, Q, the process for $f(t,T)$ can be written as

$$df(t,T) \equiv \mu_{T^Q}(t)dt + \sigma_T(t)dZ_{T^Q}(t) \tag{56}$$

with a correlation structure, represented by a function $c(\bullet)$:

$$\text{corr}[dZ_{T_1}^Q(t), dZ_{T_2}^Q(t)] \equiv c(t, T_1, T_2). \tag{57}$$

It can be shown that, similarly to the HJM models, the risk-neutral drift is completely determined by the volatility and correlation structure, namely,

$$\mu_T^Q(t) = \sigma_T(t)\int_t^T \sigma_s(t)c(t,T,s)ds. \tag{58}$$

Note that when $c(\bullet)$ is unity, this model recovers the original HJM formulation. So, this infinite-dimensional formulation captures the correlations between forward rates (hence, bond prices) of differing maturities in a parsimonious way. The model will thus improve the effectiveness of hedging strategies that use comparable bonds.

Pricing bond options in the infinite-dimensional models is surprisingly straightforward. If the volatility and correlation structures are deterministic, as in Kennedy's (1994, 1997) Gaussian random-field model, the bond prices are distributed lognormally and the option prices can be written in a Black–Scholes form. If the volatility and/or correlation structures are stochastic, closed-form formulas are not attainable but numerical calculation of the option prices is feasible. In particular, one may apply the technique of characteristic functions fashioned by Heston (1993) to express the option prices in an integral form to facilitate fast numerical calculation, as demonstrated in Goldstein. Santa-Clara and Sornette used their stochastic-string model to price the delivery option embedded in long-bond futures contracts. In addition, Longstaff, Santa-Clara, and Schwartz (1999) compared the valuation and optimal exercise of American-style swaptions in a multifactor stochastic-string model with the standard one-factor and multifactor models and found that the standard models significantly undervalue the American-style swaptions and lead to suboptimal exercise strategies and biased hedge ratios.

Conclusion

Tremendous advances have been made in recent years in the development of term-structure models and their applications in pricing interest rate derivative products. The progress is rooted in a deepening understanding of the financial markets by academic researchers, increasing knowledge of stochastic processes, ready availability of computational power, and constant innovation in financial products in the financial industry. This article aimed to provide a coherent introduction to the major recent developments in modeling the term structure and pricing of fixed-income derivatives. The breadth and sophistication of these models require much more space, however, than afforded by an article. I have not even touched upon important issues of estimation, calibration, and numerical implementation of these models, some of which issues are discussed elsewhere.[38]

On the one hand, the models presented here have some failings. They assert, either explicitly or implicitly, that markets are complete—that is, that the risks associated with interest rate derivatives are spanned by bonds in the market and hence can be hedged by bonds in the market. Jumps and stochastic volatility in the interest rate process cast doubts, however, on this assertion. In fact, Collin-Dufresne and Goldstein (2001) provided evidence from the swap market that fails to support the assertion. If their finding proves to be pervasive in further empirical studies, models for derivative pricing that go beyond the established pricing-by-replication approach are needed.

On the other hand, the implementation of term-structure models has stimulated further development of models that are (1) more directly applicable with observed rates, as in the case of the market models, and (2) more internally consistent through the incorporation of an adequate correlation structure, as in the case of the infinite-dimensional models. Future progress in merging these approaches[39] and in developing new ways of thinking about these issues will certainly yield fruitful rewards, both intellectual and financial.

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Notes

1. Tuckman (1995) provides an accessible treatment; the approaches of Hull (2000), Hunt and Kennedy (2000), James and Webber (2000), Martellini and Priaulet (2001), Musiela and Rutkowski (1997), and Rebonato (1998), among others, require varying degrees of mathematical sophistication. Earlier overviews of term-structure models include Back (1996), Flesaker and Hughston (1997), Ho (1995), Hughston (1996), and Marsh (1995).
2. Although term-structure modeling is a highly mathematical undertaking, I have tried to focus on the intuition by keeping only the necessary mathematical formulas in the main text and placing descriptive discussions and the more technical content in footnotes.
3. The term structure of interest rates is also called "the yield curve of zero-coupon bonds." With this correspondence in mind, I use "term structure" and "yield curve" interchangeably in this article. In market jargon, however, the yield curve may refer to yields to maturity of on-the-run coupon bonds. The next section specifies more exactly the definition of term structure discussed in this article.
4. A straight bond is a bond with fixed coupon and principal payments and without any optional provisions.
5. The empirical estimation of term-structure models is discussed by Chapman and Pearson in this issue of the *Financial Analysts Journal*.
6. As Back pointed out, this classification may be a misuse of terms, because equilibrium models like the CIR model do not admit arbitrage in the economic environment specified in the model. Moreover, arbitrage models, such as the Hull–White (1990) model, are constructed by making the coefficients in equilibrium models (e.g., the Vasicek model) time varying.
7. I do not discuss bond characteristics, such as duration and convexity, or portfolio strategies, such as immunization. For descriptions of the state of the art in both cases, interested readers can consult the finance textbooks or Fabozzi and Fong (1994).
8. An introduction to stochastic calculus and continuous-time formulation in finance can be found in the appendixes of previously cited books or in, among others, Karatzas and Shreve (1991) and Baxter and Rennie (1996).
9. Since 1984, investors have been able to hold strips of individual coupons and principals of U.S. T-bonds and trade them separately. These securities are called STRIPS (for "separate trading of registered interest and principal of securities") and behave like zero-coupon bonds. After 1987, coupon bonds could also be reconstructed from STRIPS.
10. In fact, the recently developed market models are specifically designed to describe the term structure of LIBORs or swap rates, as discussed in the "Market Model" section of this article. For a discussion of the term structure of swap spreads, see He (2000).
11. I am not considering the effects of trading costs, liquidity, margin constraints, and/or other market frictions.
12. A European option can be exercised only at the end of its life, on its expiration date.
13. The risk-neutral measure can be shown to be an equivalent martingale measure with respect to the subjective real-world probability measure. Two probability measures are said to be equivalent if an event having a nonzero probability of occurring under one measure is also to occur under the other measure, albeit with a different nonzero probability. The risk-neutral valuation methodology is perhaps the most important concept in derivative pricing, but it is difficult to grasp at first. Tuckman provided an excellent and accessible exposition of this methodology through a binomial tree construction to price a call option on a bond.
14. An affine model is one in which zero-coupon yields are linear with respect to underlying state variables (see Duffie and Kan 1996).
15. See Chapman and Pearson (2001) for a detailed discussion.
16. By Ito's Lemma from stochastic calculus, $\mu_P(t,T)P(t,T) = [(\partial P/\partial t) + (\partial P/\partial r)\mu(r)] + (1/2)(\partial^2 P/\partial r^2)\,\sigma^2(r)$ and $\sigma_P(t,T)P(t,T) = (\partial P/\partial r)\sigma(r)$.

17. By applying Ito's Lemma, the bond price, $P(t,T)$, satisfies a partial differential equation: $(1/2)(\partial^2 P/\partial r^2)\sigma^2(r) + (\partial P/\partial r)[\mu(r) - \lambda(r)\sigma(r)] + (\partial P/\partial t) - rP = 0$, with boundary condition $P(T,T) = 1$. This approach can be used to solve for other contingent-claim prices, including prices of fixed-income derivatives.
18. The mathematical expressions for $A(\tau)$ and $B(\tau)$ in the Vasicek model are

$$A(\tau) = [B(\tau) - \tau]\left[\bar{r} - (\lambda_0\sigma/\kappa) - 1/2(\sigma/\kappa)^2\right] - \left[\sigma^2 B(\tau)^2\right]/4\kappa$$

and $B(\tau) = (1 - e^{-\kappa\tau})/\kappa$.
19. Jamshidian (1989) provided a formula for pricing European options on zero-coupon bonds as well as on coupon bonds in the Vasicek model. Given that the process for the bond price is conditionally lognormal, the price of a European option has a Black–Scholes-like formula. The time-0 price of a call option on a discount bond maturing at T and expiring at $T' < T$ is given as $FP(0,T)N(d_+) - KP(0,T')N(d_-)$, where F is the face value of the bond, K is the strike price,

$$d_{\pm} = \frac{\ln[FP(0,T)/KP(0,T')] \pm \sigma_P^2/2}{\sigma_P}, \text{ and}$$

$$\sigma_P = \frac{\sigma\left[1 - e^{-\kappa(T-T')}\right]}{\kappa}\sqrt{\frac{1 - e^{-2\kappa T'}}{2\kappa}}.$$

20. The expressions for $A(\tau)$ and $B(\tau)$ in the CIR model are

$$A(\tau) = \frac{2\kappa\bar{r}}{\sigma^2}\ln\left[\frac{2\gamma e^{(\kappa+\lambda_0+\gamma)\tau/2}}{2\gamma + (\kappa+\lambda_0+\gamma)(e^{\gamma\tau}-1)}\right] \text{ and}$$

$$B(\tau) = \frac{2(e^{\gamma\tau}-1)}{2\gamma + (\kappa+\lambda_0+\gamma)(e^{\gamma\tau}-1)}, \text{ where}$$

$$\gamma = \sqrt{(\kappa+\lambda_0)^2 + 2\sigma^2}.$$

21. A console is a bond of infinite maturity that pays out periodic coupons. It can be thought of as an annuity that never matures. In the Brennan–Schwartz model, the console bond pays a continuous coupon at the annual rate of c and $l(t)$ is the continuously compounded console yield, sometimes called "the long yield."
22. The model actually has three state variables, but one of them is not relevant to the dynamics of the short rate.
23. Another class of models that has received attention recently is the quadratic models, in which yields are quadratic functions of the factors. Longstaff's (1989) nonlinear term-structure model, its modification by Beaglehole and Tenney (1992), and Constantinides' (1992) SAINTS model for nominal term structure are special cases. Ahn, Dittmar, and Gallant (forthcoming) characterize and empirically estimate some of these models.
24. See Chapman and Pearson for a discussion of recent studies concerning the choice of market price of risk and the empirical performance of affine models.
25. American options can be exercised on any business day after purchase through the expiration of the option.
26. See Hull for an excellent exposition of the models and references for implementing them.
27. The complete market assumption effectively states that all payoffs in the market can be replicated by combinations of traded securities. These conditions imply the existence of a unique equivalent martingale measure. This equivalent martingale measure, Q, can be constructed from a set of N different discount bonds. Heath, Jarrow, and Morton assumed that there are N-basis bonds and that other bonds are priced as linear combinations of these basis bonds. Here, the market prices of risk, λ_i, need not be directly specified because the information is already contained in the market prices of the N-basis bonds.
28. Under risk-neutral measure Q,

$$df(t,T) = \alpha^Q(t,T)dt + \sum_{i=1}^N \sigma_i(t,T)dW_i^Q(t), \text{ where}$$

$$\alpha^Q(t,T) = \alpha(t,T) + \sum_{i=1}^N \lambda_i(t)\sigma_i(t,T). \text{ Then,}$$

$$\alpha^Q(t,T) = \sum_{i=1}^N \sigma_i(t,T)\int_t^T \sigma_i(t,s)ds.$$

29. Some recent papers tackle the problem of valuing American options by using the Monte Carlo simulation in innovative ways. See, for example, Longstaff and Schwartz (2001).
30. Note that in the CIR model, interest rates remain positive as long as $2\kappa\theta(t) > \sigma^2$. Meeting this restriction cannot be guaranteed when $\theta(t)$ is fitted to match current bond prices.
31. The positive-interest-rate approach of Flesaker and Hughston has a close connection to the "potential" method of Rogers (1997). For further details, see Flesaker and Hughston (1996, 1997), Hunt and Kennedy, Musiela and Rutkowski, Rutkowski (1997), and Jin and Glasserman (2001).
32. The Black model is based on the model Fischer Black proposed for valuing European options on commodity futures contracts. In valuing interest rate derivatives, such as bond caps and swaptions, the model assumes that the underlying bond price or interest rate at the option expiration is distributed lognormally and the expected future price or rate is its forward price or rate. A Black–Scholes-type formula is then obtained for various European bond options, including caps, floors, and swaptions.
33. Note that Equation 3b defines a continuously compounded forward rate, $f(t,T,\delta)$.
34. The existence of a unique nonnegative solution to the stochastic differential equation (Equation 49) under suitable regularity conditions is proven in Brace, Gatarek, and Musiela. Miltersen, Sandmann, and Sondermann showed that the lognormal assumption for forward rates of finite maturities is consistent with the HJM framework for a specific choice of volatility structure.
35. Jamshidian (1997) also discussed the valuation of options that depend on both LIBOR and swap rates, such as spread options and LIBOR trigger swaps, in the framework of the market model.
36. In Santa-Clara and Sornett, the shocks are indexed by time-to-maturity, $T - t$. Although a formulation that uses time-to-maturity indexing has some advantages, I kept the maturity indexing to keep notations in this article consistent.
37. "Brownian field" and "string shock" refer to the same construction of two-dimensional stochastic processes. Santa-Clara and Sornette provided a general characterization of the properties that these processes need to satisfy and showed a few admissible and nonadmissible examples. Goldstein demonstrated different constructions of these processes to meet various smoothness requirements for forward curves. A rigorous treatment of infinite-dimensional models is in Filipović.
38. For example, Chapman and Pearson discuss empirical issues relating to short-rate dynamics and the estimation of term-structure models.
39. Recent work by Longstaff, Santa-Clara, and Schwartz represent such an effort.

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