EXPECTED DRAWDOWNS

CENGIZ Y. BELENTEPE

ABSTRACT. How does one estimate what an investment's expected drawdown will be by using just the average and standard deviation of the investment's performance? Can one use diversification to reduce the expected drawdown of one's investments? In this article, we address and answer both of these questions. In fact, we provide several practical ideas and formulas which may assist one in the active management of their portfolios.

1. Motivation

One only need spend a short amount of time in the asset management industry to realize that the concept of drawdown is paramount to investors. The *current drawdown* of an investment with price process $\{P_t : 0 \leq t \leq T\}$ is defined by

$$D_t \quad = \quad 1 - P_t / \max_{0 \leq s \leq t} P_s,$$

and D_t is a quantity of great concern to investors. It represents the fraction of one's wealth that has been lost since the investment was at its peak. In some ways, it reflects how much regret we have for not having exited our investment at an earlier, more fortuitous time.

To gain insight into the properties of drawdown, we consider the simplest investment model, geometric Brownian motion. We then consider two tasks. Our first task is to calculate the expected current drawdown of our investment which will provide investors with a benchmark for drawdown risk. Our second task is to see how diversification can help control the expected current drawdown.

2. Distributions and densities

Before making any probabilistic statements about drawdowns, we must first define the stochastic differential equation that describes our investment model. Letting μ denote a constant drift rate and σ^2 denote an instantaneous variance, we assume our investment P_t satisfies the stochastic differential equation

$$dP_t \quad = \quad \mu P_t dt + \sigma P_t dB_t$$

where B_t is standard Brownian motion. The solution to this stochastic differential equation is known to be

$$P_t \quad = \quad P_0 \exp\left\{ \left(\mu - \frac{1}{2}\sigma^2 \right) t + \sigma B_t \right\}$$

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and without loss of generality we assume our initial investment is \$1, i.e. $P_0 = 1$. Letting

$$\begin{aligned} X_t &= rt + \sigma B_t, \qquad r \stackrel{\text{def}}{=} \mu - \frac{1}{2}\sigma^2 \\ Y_t &= \max\{X_s : s \in [0,t]\}, \end{aligned}$$

we can rewrite D_t as

$$D_t = 1 - \frac{e^{X_t}}{e^{Y_t}} = 1 - e^{X_t - Y_t}.$$

From the reflection principle for Brownian motion the joint distribution of (X_t, Y_t) when $y \geq x$ and $y \geq 0$ is known to be

$$P(X_t \leq x, Y_t \leq y) = \Phi\left(\frac{x - rt}{\sigma\sqrt{t}}\right) - \exp\left\{\frac{2ry}{\sigma^2}\right\}\Phi\left(\frac{x - 2y - rt}{\sigma\sqrt{t}}\right),$$

as one finds, for example, in Harrison ([1], pp 22). Differentiating twice, one has the joint density of (X_t, Y_t) as

$$f_{X_t,Y_t}(x,y) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(\frac{2ry}{\sigma^2} - \frac{(x-2y-rt)^2}{2\sigma^2 t}\right)\left[\frac{2(2y+rt-x)}{\sigma^2 t} - \frac{2r}{\sigma^2}\right]$$

which allows us to compute $P(D_t \geq 1 - \alpha)$ for $0 \leq \alpha \leq 1$. In fact, we get a nice formula for calculating the probability of a drawdown exceeding a specific level

$$\begin{aligned} P(D_t \geq 1-\alpha) &= P(1 - e^{X_t - Y_t} \geq 1 - \alpha) = P(e^{X_t - Y_t} \leq \alpha) \\ &= 1 - P(Y_t - X_t \leq -\log\alpha) \\ &= 1 - \int_0^\infty \int_{y+\log\alpha}^{y} \frac{\exp\left(\frac{2ry}{\sigma^2} - \frac{(x-2y-rt)^2}{2\sigma^2 t}\right)}{\sqrt{2\pi\sigma^2 t}}\left[\frac{2(2y+rt-x)}{\sigma^2 t} - \frac{2r}{\sigma^2}\right] dxdy \\ &= 1 - \Phi\left(\frac{-\log\alpha + rt}{\sigma\sqrt{t}}\right) - \alpha^{\frac{2r}{\sigma^2}}\left[\Phi\left(\frac{-\log\alpha - rt}{\sigma\sqrt{t}}\right) - 1\right]. \end{aligned}$$

3. Expected drawdown percentage

Noticing that $E(D_t) = 1 - E(e^{X_t - Y_t})$, we use the distribution for $e^{X_t - Y_t}$ that we just calculated to derive its density and integrate for $E(D_t)$. The density of $e^{X_t - Y_t}$ is

$$\begin{aligned} f(\alpha) &= \frac{d}{d\alpha} P(e^{X_t - Y_t} \leq \alpha) \\ &= \frac{1}{\alpha\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{(-\log\alpha + rt)^2}{2t\sigma^2}\right) + \frac{\alpha^{\frac{2r}{\sigma^2}-1}}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{(\log\alpha + rt)^2}{2t\sigma^2}\right) \\ &\quad + \frac{2r}{\sigma^2}\alpha^{\frac{2r}{\sigma^2}-1} - \frac{2r}{\sigma^2}\alpha^{\frac{2r}{\sigma^2}-1}\Phi\left(\frac{-\log\alpha - rt}{\sigma\sqrt{t}}\right), \end{aligned}$$

so

$$
\begin{aligned}
E(D_t) &= 1 - E(e^{X_t}/e^{Y_t}) = 1 - \int_0^1 \alpha f(\alpha) d\alpha \\
&= 1 - \frac{2r\Phi\left(\frac{r\sqrt{t}}{\sigma}\right) + 2\exp\left(rt + \frac{\sigma^2 t}{2}\right)\left(r+\sigma^2\right)\Phi\left(-\frac{\sqrt{t}(r+\sigma^2)}{\sigma}\right)}{2r+\sigma^2}. \qquad (3.1)
\end{aligned}
$$

This formula for $E(D_t)$ permits an investor to relate μ, σ and t to a drawdown percentage by recalling that r is defined as $\mu - \sigma^2/2$. Likewise, carrying out similar calculations, the variance of D_t can be shown to be

$$
\begin{aligned}
\mathrm{Var}(D_t) &= \frac{2r\Phi\left(\frac{r\sqrt{t}}{\sigma}\right) + 2\exp\left(2rt + 2\sigma^2 t\right)\left(r+2\sigma^2\right)\Phi\left(-\frac{\sqrt{t}(r+2\sigma^2)}{\sigma}\right)}{2(r+\sigma^2)} \\
&\quad - \left(\frac{2r\Phi\left(\frac{r\sqrt{t}}{\sigma}\right) + 2\exp\left(rt + \frac{\sigma^2 t}{2}\right)\left(r+\sigma^2\right)\Phi\left(-\frac{\sqrt{t}(r+\sigma^2)}{\sigma}\right)}{2r+\sigma^2}\right)^2.
\end{aligned}
$$

A natural question after looking at Equation (3.1) is what happens as we let t approach infinity. Using the following bounds on the tail probability of a Gaussian random variable Z

$$
\frac{1}{\sqrt{2\pi}}\frac{1}{z+1/z}\exp\left(-\frac{z^2}{2}\right) \le \quad \Phi(-z) \quad \le \frac{1}{\sqrt{2\pi}}\frac{1}{z}\exp\left(-\frac{z^2}{2}\right), \; z \ge 0.
$$

we can answer this if care is taken to consider three distinct regions : $r+\sigma^2 \le 0,\; 0 \le r+\sigma^2 \le \sigma^2, r > 0$. Making use of the preceding inequality when relevant we have

$$
\begin{aligned}
\lim_{t\to\infty} E(D_t) &= 1 - \lim_{t\to\infty} \frac{2r\Phi\left(\frac{r\sqrt{t}}{\sigma}\right) + 2\exp\left(rt + \frac{\sigma^2 t}{2}\right)\left(r+\sigma^2\right)\Phi\left(-\frac{\sqrt{t}(r+\sigma^2)}{\sigma}\right)}{2r+\sigma^2} \\
&= \begin{cases} 1 & \mu \le \sigma^2/2 \\ \frac{\sigma^2}{2r+\sigma^2} = \frac{\sigma^2}{2\mu} & \mu > \sigma^2/2. \end{cases}
\end{aligned}
$$

Figure (1) displays the expected drawdown for various levels of σ and t and exhibits this limiting behavior. A similar calculation shows that

$$
\lim_{t\to\infty} \mathrm{Var}(D_t) = \begin{cases} 0 & \mu \le \sigma^2/2 \\ \frac{r}{r+\sigma^2} - \left(\frac{2r}{2r+\sigma^2}\right)^2 & \mu > \sigma^2/2. \end{cases}
$$

4. Controlling drawdown

What are the implications of this for an investor faced with the opportunity to invest in any of k independent investments? In particular, consider k independent investments, each of which follow geometric Brownian motion with constant drift rate μ and instantaneous variance σ^2. Letting $P_{i,t}$ denote an investment in the i^{th} investment, we assume each of our investments $P_{i,t}$ satisfy the stochastic differential equation

$$
dP_{i,t} = rP_{i,t}dt + \sigma P_{i,t}dB_{i,t}, \quad i = 1, \ldots, k
$$

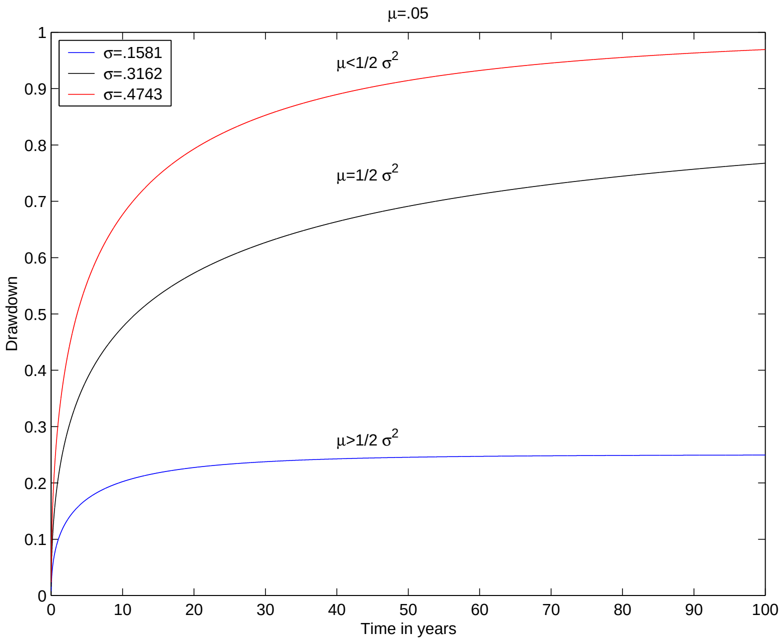

FIGURE 1. Expected drawdown $E(D_t)$ and dependence on σ and t.

where $B_{i,t}$ are standard Brownian motions independent of one another.

Within this setup, an investor who decides to buy and hold $1/k^{\text{th}}$ of his wealth in each of the k investments would have the following wealth process at time t

$$W_t = \frac{1}{k}\sum_{i=1}^{k} P_{i,t}.$$

At which point we could easily be stumped. After all, it is an unfortunate fact that the arithmetic average of k geometric Brownian motions is not itself a geometric Brownian motion. Practitioners of continuous time finance have been presented with this very same problem under the guise of pricing index options. Index options are nothing more than the option to buy or sell an arithmetic weighted portfolio of individual assets at a certain strike price. Their solution is the following – match moments between the true process W_t and a geometric Brownian motion W_t' meant to approximate the true process and solve for the r' and σ'^2 of this new geometric Brownian motion. It turns out that solving the system of equations consequent from

$$\begin{aligned} E(W_t) &= E(W_t') \\ \exp\left(rt + \frac{\sigma^2 t}{2}\right) &= \exp\left(r't + \frac{\sigma'^2 t}{2}\right) \end{aligned}$$

and

$$\begin{aligned}
\mathrm{Var}(W_t) &= \mathrm{Var}(W'_t) \\
\frac{1}{k}\,\mathrm{Var}(P_t) &= e^{2r't+\sigma'^2 t}\left(e^{\sigma'^2 t}-1\right) \\
\frac{1}{k}\,e^{2rt+\sigma^2 t}\left(e^{\sigma^2 t}-1\right) &= e^{2r't+\sigma'^2 t}\left(e^{\sigma'^2 t}-1\right)
\end{aligned}$$

leads to the parameters of our new geometric Brownian motion W'_t as

$$\begin{aligned}
\sigma'^2 &= \log\left(1-\frac{1}{k}+\frac{e^{\sigma^2}}{k}\right) \\
r' &= \frac{2r+\sigma^2-\log\left(1-\frac{1}{k}+\frac{e^{\sigma^2}}{k}\right)}{2}
\end{aligned}$$

Now in order to see the expected percentage drawdown from investing in k independent investments, plug these values of r' and σ'^2 back into Equation (3.1). In other words, $E(D_{k,t})$ equals

$$1-\frac{2r'\Phi\left(\frac{r'\sqrt{t}}{\sigma'}\right)+2\exp\left(r't+\frac{\sigma'^2 t}{2}\right)\left(r'+\sigma'^2\right)\Phi\left(-\frac{\sqrt{t}(r'+\sigma'^2)}{\sigma'}\right)}{2r'+\sigma'^2}. \tag{4.1}$$

where $D_{k,t}$ is the drawdown percentage when we buy and hold an equal fraction $1/k$ in each of our k independent investments.

5. Simulations

The moment matching method is short on formal justifications, but by simulations one finds that it performs rather well. Table (1) gives the average percentage drawdown and the standard deviation of the drawdown from 10,000 simulations with $r = .0004, \sigma = .01897$ and $t = 2500$ as well as the values of $E(D_{k,2500})$ derived from Equation (4.1). This choice of r, σ and t is the daily equivalent of 10% annual return and 30% annual volatility over a ten year period. Figure (2) displays the histogram of drawdown percentages from this simulation.

The table informs us that the investor with 8 investments as opposed to only 1 investment achieves a nearly 80% reduction in his expected drawdown. Moreover, he achieves a 70% reduction in the standard deviation of this drawdown. So not only does he have lower drawdowns, but he also has less uncertainty as to what these drawdowns will be. Clearly, diversification has a powerful impact on one's portfolio.

k	Simulation $\bar{D}_{k,2500}$	Moment Approximated $E(D_{k,2500})$
1	.27	.28
2	.17	.15
4	.10	.07
8	.06	.04

k	Simulation $\mathrm{S}_{D_{k,2500}}$	Moment Approximated $\sigma_{D_{k,2500}}$
1	.21	.21
2	.14	.13
4	.09	.07
8	.06	.04

Table 1. Comparing the mean and standard deviations of $D_{k,t}$ from simulation and approximation.

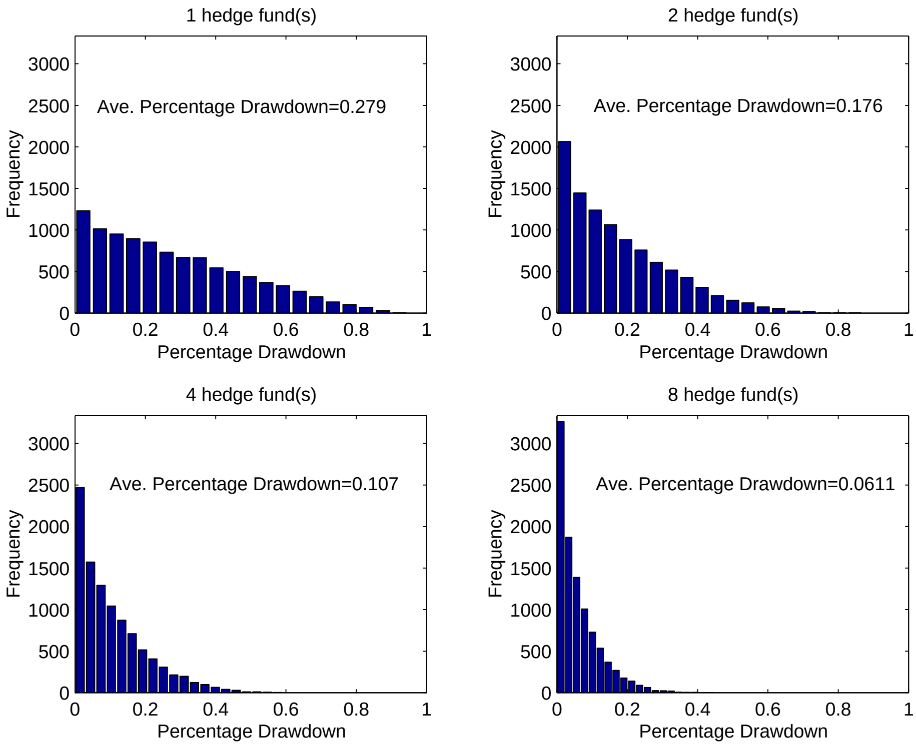

Figure 2. Histogram of simulation

6. Conclusion

The implications of this are clear - drawdown can be minimized by having multiple strategies on one asset or one strategy on multiple assets. Due to the assumptions we have made to calculate expected drawdowns, one should use our results in the following manner. Find how many strategies are required to give you your desired expected current drawdown $E(D_{k,t})$ and realize this number, k, will most likely be the *minimum* number of strategies required in real life.

References

[1] Harrison, J.M., *Brownian Motion and Stochastic Flow Systems*, Wiley, New York, 1985.

[2] Steele, J.M., *Stochastic Calculus and Financial Applications*, Springer-Verlag, New York, 2000.

Department of Statistics, Wharton School, University of Pennsylvania, Huntsman Hall, Philadelphia PA 19104

E-mail address: cengiz@wharton.upenn.edu