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NONCLASSICAL SHOCKS AND KINETIC RELATIONS: STRICTLY HYPERBOLIC SYSTEMS*

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Abstract. We consider strictly hyperbolic systems of conservation laws whose characteristic fields are not genuinely nonlinear, and we introduce a framework for the nonclassical shocks generated by diffusive or diffusive-dispersive approximations. A nonclassical shock does not fulfill the Liu entropy criterion and turns out to be undercompressive.

We study the Riemann problem in the class of solutions satisfying a single entropy inequality, the only such constraint available for general diffusive-dispersive approximations. Each non-genuinely nonlinear characteristic field admits a *two-dimensional wave set*, instead of the classical one-dimensional wave curve. In specific applications, these wave sets are narrow and resemble the classical curves. We find that even in strictly hyperbolic systems, nonclassical shocks with arbitrarily small amplitudes occur. The Riemann problem can be solved uniquely using nonclassical shocks, provided an additional constraint is imposed: we stipulate that the entropy dissipation across any nonclassical shock be a given constitutive function. We call this admissibility criterion a *kinetic relation*, by analogy with similar laws introduced in material science for propagating phase boundaries. In particular, the kinetic relation may be expressed as a function of the propagation speed. It is derived from traveling waves and, typically, depends on the ratio of the diffusion and dispersion parameters.

Key words. conservation laws, hyperbolic entropy, shock wave, kinetic relation, nonclassical shock

AMS subject classifications. 35L65, 76L05

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1. Introduction.

In this paper, we consider discontinuous solutions to hyperbolic systems of conservation laws that do not fulfill the classical entropy criteria, carrying over to systems the discussion we initiated in [22] for scalar equations with nonconvex fluxes. We develop a framework for the existence and uniqueness of the *nonclassical* shock waves that arise as limits of diffusive-dispersive approximations. It is natural to constrain the solutions to the hyperbolic system with an entropy inequality for a *single*, strictly convex entropy pair. This condition is weaker than the Liu [41] entropy criterion.

A *nonclassical shock* is defined as one that does not satisfy the Liu criterion. It turns out that such a shock is *undercompressive*: the number of characteristics impinging on the discontinuity is smaller than that imposed by the (classical) Lax shock inequalities. Such waves are underdetermined (in the sense of linear analysis) and sensitive to the form of the diffusive-dispersive mechanism.

The focus of this work is on strictly hyperbolic systems where one (or more) characteristic field lacks genuine nonlinearity, such as those describing the dynamics of elastic materials or magnetic fluids. A key observation is that undercompressive shocks may arise for such systems through balanced diffusive and dispersive mechanisms: this

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is the case even for shocks having *arbitrary small* amplitude. We concentrate here on the Riemann problem which is fundamental in the theory of conservation laws. A typical Riemann solution combines classical (shock and rarefaction) waves and nonclassical shocks. The numerical analysis of nonclassical shocks is investigated in a companion paper [23].

We build here upon extensive activity on undercompressive waves for nonstrictly hyperbolic systems and systems with change of type. In the examples studied in the literature, the undercompressive waves have finite strength; they were found to be necessary in order to solve the Riemann problem and, therefore, reflect a property of the flux-function of the system. We refer the reader to Azevedo et al. [3], Freistühler [18], Isaacson, Marchesin, and Plohr [28], Isaacson et al. [27], Keyfitz [31], Liu and Zumbrun [46, 47], Schecter and Shearer [52], Slemrod [59], and the references therein.

The basic concepts and the analysis of the traveling waves associated with such nonstandard discontinuities and a resolution of the Riemann problem for some mathematical models can be also found in [28, 32, 44, 45, 58]. The large-time asymptotic stability of under- or overcompressive shocks (the number of impinging characteristics in the latter is larger) is proven in [19, 43, 46, 47]. Liu and Zumbrun observe [47] that, for undercompressive shocks, the asymptotic state for large times cannot be determined solely from the mass of the initial perturbation, but must also take into account the diffusive effects of a parabolic augmented system of equations.

Several examples from continuum mechanics are known to exhibit undercompressive shocks. The system of magnetohydrodynamics lacks both genuine nonlinearity and strict hyperbolicity (Brio and Wu [5]). It has been observed numerically, as well as analytically, that nonstandard shock waves not fulfilling the classical entropy criteria arise with certain approximations.

MHD shocks may be either undercompressive or overcompressive. Those shocks are called nonstandard or intermediate in the MHD literature and are critical to the understanding of important phenomena such as the effect of the solar wind (Wu [63]). For various results on the Riemann problem for a rotationally invariant model in MHD, we refer to [4, 6, 8, 17, 32, 65]. See also [21] for another model. There is also an extensive literature on phase boundaries in materials admitting phase transformations of the austenite-martensite type. When the stress-strain relation for a material is decreasing on an interval, the system of elastodynamics is of the hyperbolic-elliptic type. Propagating phase boundaries are still another example of undercompressive waves. They are fundamental to understanding phase transformation processes. See [15, 26, 56, 57, 59] as well as [1, 2, 38, 61, 62]. See also [48] for a general review on the nonlinear waves arising in fluids and materials, with or without phase transitions.

A pioneering study of the effect of vanishing diffusion and dispersion terms in scalar conservation laws can be found in Schonbek [53] using the compensated compactness method. She proved a convergence theorem toward weak solutions. LeFloch and Natalini [39] used the concept of measure-valued solution and established convergence results assuming that the diffusion dominates the dispersion.

The works by Wu [64] and Jacobs, McKinney, and Shearer [30] established the first existence result of undercompressive shocks for the modified KdV–Burgers equation and motivated us in [22].

The present series of papers [22, 23] is intended as a contribution toward unifying ideas behind some of the above works. We pursue a better understanding of simple models giving rise to undercompressive shocks. Deriving entropy criteria for their selection is one of the main challenges in the field. The classical criteria developed by Dafermos [10, 11, 12], Lax [34, 35], Liu [41, 42], Oleinik [50], etc., cannot be applied directly. In contrast to previous works, we focus here primarily on strictly hyperbolic systems having nongenuinely nonlinear characteristic fields.

Given a strictly convex entropy pair, we first endeavor to describe the set of all solutions to the Riemann problem that satisfy a single entropy inequality. Allowing nonclassical shocks leads to a lack of uniqueness for the Riemann problem and a *multi-parameter family* of solutions can be constructed. Our construction is an extension to Liu's theorem on the resolution of the Riemann problem which was based on what is now called the Liu criterion. This analysis provides a complete description of all the Riemann solutions generated by any diffusive-dispersive approximation compatible with a given entropy pair (section 2). We observe that characterizing limits of approximate sequences of solutions to hyperbolic systems via pointwise relations on the propagating discontinuities in the limiting solution may not be possible in the most general situation (see, for instance, Glimm [20] and LeFloch and Tzavaras [40]). In this regard, our analysis is pertinent toward describing the *set* of all possible such limits. In our presentation, pointwise constraints are added afterward.

Next we investigate a way of selecting a unique nonclassical solution. We propose to make the selection based on the entropy dissipation, which is a fundamental quantity from both mathematical and physical standpoints. We stipulate that the entropy dissipation of a nonclassical shock be a given function, the "kinetic function." It may be assumed, for instance, that the kinetic function depends only on the *speed of the nonclassical shock*. We call such an admissibility criterion a *kinetic relation* by analogy with similar laws introduced in material science.

Therefore this generalizes to strictly hyperbolic systems the notion of kinetic relation known for the hyperbolic-elliptic system of phase transitions (Abeyaratne and Knowles [1, 2] and Truskinovsky [61, 62]; see also LeFloch [38]) and for nonconvex scalar conservation laws (Hayes and LeFloch [22] and Kulikovsky [33]). The paper by Truskinovsky [62] includes a review of these issues in material science.

In section 2, we construct a unique solution to the Riemann problem in the class of nonclassical solutions when the kinetic relation is enforced. For some Riemann data choosing between the classical solution and the nonclassical one may be still necessary (see section 2). When a specific augmented system including diffusive/dispersive effects is provided, the entropy dissipation and therefore the kinetic function can be determined. Small-scale effects neglected in the mathematical modeling at the hyperbolic level are essential to understanding the behavior of nonclassical shocks. The kinetic function can be obtained from the equation of the traveling wave solutions associated with the diffusive-dispersive model.

Classical and nonclassical shock are very different in nature. The classical shocks are associated with the continuum spectrum of the traveling wave equation and the nonclassical shocks with its discrete spectrum. Typically, given a (left) state, and restricting attention to a given wave family, there exists a one-parameter family of right states that can be attained with a classical shock, but a single right state can be attained by a nonclassical shock.

In several systems arising in the applications in continuum mechanics, the entropy dissipation is related to the total energy and may be viewed as a force driving the propagation of the nonclassical propagating discontinuities. We also consider here the Riemann problems with large amplitude for two specific examples of interest: a system from nonlinear elastodynamics based on a nonconvex strain-stress law, which is a strictly hyperbolic system with two nongenuinely nonlinear fields (sections 3 and 4), and a model from magnetohydrodynamics, which has an umbilic point and one linearly degenerate characteristic field (section 5). In these examples we demonstrate numerically that certain diffusive-dispersive approximations generate nonclassical shocks.

The kinetic relation may be used in the design of a numerical scheme consistent with the underlying regularization, avoiding the (costly) resolution of small-scale effects. Hou, LeFloch, and Rosakis [25] proposed recently, for computing propagating phase boundaries in a two-dimensional plate, a consistent method based on the level set formulation. For difference schemes generating nonclassical shocks, one can consult [23, 24].

2. A framework for nonclassical shocks in systems.

2.1. Preliminaries. Here we shall motivate the definition of nonclassical solution. Consider a system of hyperbolic conservation laws:

$$\partial_t u + \partial_x f(u) = 0, \qquad u(x,t) \in \mathcal{U}, \tag{2.1}$$

where $\mathcal{U}$ is a convex and open subset of $\mathbb{R}^N$ and the flux-function $f : \mathcal{U} \to \mathbb{R}^N$ is a smooth mapping. We assume that the system is endowed with a strictly convex entropy pair (U, F); that is, $\nabla F^T = \nabla U^T Df$ and $\nabla^2 U(u) \geq C\, Id$ with $C > 0$. This, in particular, implies that the system is hyperbolic, although not necessarily strictly hyperbolic.

Suppose that the "good" solutions to (2.1) according to some underlying physical interpretation are to be obtained as limits of a diffusive-dispersive approximation scheme of the form

$$\partial_t u_\epsilon + \partial_x f(u_\epsilon) = \epsilon\, \partial_x\big(B_1(u_\epsilon)\partial_x u_\epsilon\big) + \epsilon^2\, \partial_x\big(B_2(u_\epsilon)\partial_{xx} u_\epsilon\big) \tag{2.2}$$

as $\epsilon \to 0$ ($\epsilon > 0$). When B_1 and B_2 are $N \times N$ matrix-valued functions, the regularization (2.2) (together with the conditions (2.3) below) describes one large class of systems, which includes the examples in the applications we will be interested in. (The important issue of the existence of a solution u_ϵ satisfying (2.2) is out of the scope of the present paper.)

We shall say that the pair (U, F) is *compatible* with the approximation scheme (2.2) if the following conditions hold:

• The first term in the right-hand side of (2.2) is *dissipative* for the entropy U, in the sense that

$$\nabla^2 U(v) B_1(v) \text{ is a positive matrix for all } v \in \mathcal{U}. \tag{2.3i}$$

• The second term in (2.2) is *conservative* for U, in the sense that there exist $N \times N$ matrix-valued functions B_3 and B_4 such that

$$\partial_x v^T \nabla^2 U(v)^T B_2(v) \partial_{xx} v \;=\; \partial_t\big(\partial_x v^T B_3(v) \partial_x v\big) \;+\; \partial_x\big(\partial_x v^T B_4(v) \partial_x v\big) \tag{2.3ii}$$

for any solution $v : \mathbb{R} \times \mathbb{R}_+ \to \mathcal{U}$ to (2.2), and

$$B_3(v) \text{ is a nonnegative matrix for all } v \in \mathcal{U}. \tag{2.3iii}$$

Note in passing that trivial linear entropies always satisfy (2.3) but are of no use for our purpose of selecting solutions to (2.1). When (2.3) holds and $\partial_x u_\epsilon$ vanishes at infinity, one can (formally) derive from (2.2) an entropy inequality. Indeed we obtain

$$\begin{aligned}\partial_t\big(U(u_\epsilon) + {}& \epsilon^{2}\,\partial_x u_\epsilon^T B_3(u_\epsilon)\partial_x u_\epsilon\big) + \partial_x F(u_\epsilon) \\ &= \epsilon\,\partial_x\big(\nabla U(u_\epsilon)^T B_1(u_\epsilon)\partial_x u_\epsilon\big) - \epsilon\,\partial_x u_\epsilon \nabla^2 U(u_\epsilon) B_1(u_\epsilon)\partial_x u_\epsilon \\ &\quad + \epsilon^2\,\partial_x\big(\nabla U(u_\epsilon)^T B_2(u_\epsilon)\partial_{xx} u_\epsilon\big) - \epsilon^2\,\partial_x\big(\partial_x u_\epsilon B_4(u_\epsilon)\partial_x u_\epsilon\big),\end{aligned}$$

which yields the balance law

$$\begin{aligned}&\int_{\mathbb{R}} U(u_\epsilon(t))\,dx + \epsilon^2 \int_{\mathbb{R}} \partial_x u_\epsilon(t)^T B_3(u_\epsilon(t))\partial_x u_\epsilon(t)\,dx \\ &\qquad + \epsilon \int_0^t \int_{\mathbb{R}} \partial_x u_\epsilon^T \nabla^2 U(u_\epsilon) B_1(u_\epsilon)\partial_x u_\epsilon\,dxds \\ &\qquad = \int_{\mathbb{R}} U(u_\epsilon(0))\,dx + \epsilon^2 \int_{\mathbb{R}} \partial_x u_\epsilon(0)^T B_3(u_\epsilon(0))\partial_x u_\epsilon(0)\,dx\end{aligned} \tag{2.4}$$

for all $t \geq 0$ and an entropy inequality for $u = \lim_{\epsilon\to 0} u_\epsilon$ of

$$\partial_t U(u) + \partial_x F(u) \leq 0. \tag{2.5}$$

We observe that

• an arbitrary entropy for (2.1) need not be compatible with the given regularization (2.2), and the inequality (2.5) *need not hold* for an arbitrary entropy;

• the estimate (2.4) provides an a priori control on u_ϵ and its derivatives, which may be used to apply the compensated compactness method, at least if $N \leq 2$. When the latter applies the sequence u_ϵ is shown to converge to a weak solution to (2.1), (2.5). See [53, 22] and sections 4 and 5 of this paper.

As an illustration, consider the case of a scalar equation ($N = 1$) and

$$\partial_t u_\epsilon + \partial_x f(u_\epsilon) = \epsilon\,\partial_{xx} u_\epsilon + \alpha\,\epsilon^2\,\partial_{xxx} u_\epsilon, \tag{2.6}$$

where α is a real parameter. It is easily checked that the conditions (2.3) hold for $U(u) = u^2$ with $B_1 = 1$, $B_2 = \alpha$, $B_3 = \alpha/2$, and $B_4 = 0$. The estimate (2.4) reduces to

$$\int_{\mathbb{R}} u_\epsilon(t)^2\,dx + 2\,\epsilon \int_0^T \int_{\mathbb{R}} |\partial_x u_\epsilon|^2\,dxds = \int_{\mathbb{R}} u_\epsilon(0)^2\,dx, \tag{2.7}$$

and we get the inequality

$$\partial_t U(u) + \partial_x F(u) \leq 0, \qquad F'(u) := u\,f'(u).$$

Observe that, for nonquadratic entropies, (2.3) is generally violated and the inequality (2.5) does not hold, as was pointed out in Hayes and LeFloch [22].

The scaling in (2.6) is important. The diffusion dominant regularization

$$\partial_t u_\epsilon + \partial_x f(u_\epsilon) = \epsilon\,\partial_{xx} u_\epsilon + \delta\,\partial_{xxx} u_\epsilon \tag{2.8}$$

with $\delta = o(\epsilon^2)$ would bring us back to the classical theory of conservation laws, while the dispersion dominant case (2.8) with $\epsilon^2 = o(\delta)$ is the subject of the Lax–Levermore theory [36, 37]. Limiting solutions in the latter case are not weak solutions to (2.1).

This motivates us to constrain the solutions to (2.1) with the single entropy inequality (2.5). Not surprisingly, when one characteristic field (or more) of the system

(2.1) is not genuinely nonlinear, the entropy inequality will be shown to be too lax to guarantee uniqueness even for the Riemann problem. The forthcoming analysis is built upon this elementary observation.

Our analysis in [22] of the nonclassical shocks for scalar conservation laws relied on the violation of the Oleinik criterion. For systems we shall say that a shock is classical if it satisfies the Liu criterion. Definition 2.1 restates this concept.

Definition 2.1. *A propagating discontinuity is called a* nonclassical shock *when it satisfies the entropy inequality* (2.5) *but does not fulfill the Liu entropy criterion (see* (2.18) *below).* □

2.2. Nonclassical Riemann solutions. We now study the Riemann problem for nongenuinely nonlinear systems.

• Liu has constructed a unique entropy solution to the Riemann problem for such systems [41, 42]. In his construction, every shock satisfies what is now called the Liu criterion. This is described in Lemma 2.3.

• When a single entropy inequality is used, the class of admissible solutions is larger (Lemma 2.5) and undercompressive shocks are found near a curve where genuine nonlinearity breaks down (see Lemma 2.4).

• We construct a multiparameter family of solutions to the Riemann problem in Theorem 2.6. In our construction, there are two analogous cases corresponding to a minimum or a maximum of the wave speed at the point where genuine nonlinearity is lost.

This extends Liu's construction to encompass all possible limits of diffusive-dispersive approximations compatible with a given entropy pair (U, F). A further admissibility criterion will be necessary to ensure uniqueness of the entropy solution. This will be developed in subsection 2.3.

Remark 2.2. Liu's criterion is consistent with the regularization (2.2) with $B_1(u) = I$ and $B_2(u) = 0$. The latter regularization happens to be compatible with *any* convex entropy to (2.1) since, then, (2.3i) is equivalent to the convexity assumption on U and (2.3ii) and (2.3iii) are trivially satisfied. Henceforth the inequalities (2.5) in this particular case hold for *all* convex entropy pairs. However, the Liu criterion need not be satisfied by limits of more general diffusive approximations or by diffusive-dispersive ones.

We now assume that $\mathcal{U} := B(u_*, R)$ is a ball with center u_* and radius $R > 0$, and, for each u and u' in $\mathcal{U}$, the matrix $A(u, u') := \int_0^1 Df(m\,u + (1-m)\,u')\,dm$ admits N real and distinct eigenvalues $\bar\lambda_1(u, u') < \bar\lambda_2(u, u') < \cdots < \bar\lambda_N(u, u')$ and corresponding basis of right eigenvectors $\bar r_j(u, u')$ and left eigenvectors $\bar l_j(u, u')$. Throughout this section we normalize the basis so that $\bar l_j(u, u') \cdot \bar r_j(u, u') = \delta_{ij}$.

It is assumed that the wave speeds $\lambda_j(u, u')$ are strictly separated in the sense that there exist *disjoint* intervals $\big[\lambda_j^{\min}, \lambda_j^{\max}\big]$, $j = 1, 2, \dots, N$, such that

$$\lambda_j^{\min} < \bar\lambda_j(u, u') < \lambda_j^{\max} \tag{2.9}$$

for all $u, u' \in \mathcal{U}$. We also set $\lambda_j(u) := \bar\lambda_j(u, u)$, $r_j(u) := \bar r_j(u, u)$, and $l_j(u) := \bar l_j(u, u)$. When (2.1) is strictly hyperbolic, the condition (2.9) is satisfied if $\mathcal{U}$ is a sufficiently small neighborhood of u_*.

We are interested in systems admitting $N - P$ genuinely nonlinear characteristic fields and $P \le N$ nongenuinely nonlinear characteristic fields. In the latter case the scalar-valued function $u \to \nabla\lambda_j(u) \cdot r_j(u)$ does not keep a constant sign. We assume that there is a subset with P elements, $\mathbf{P} \subset \big\{1, 2, \dots, N\big\}$ such that, for $j \notin \mathbf{P}$,

$\nabla\lambda_j(u)\cdot r_j(u) > 0$ for all u (after suitable normalization of the eigenvectors), and for $j \in \mathbf{P}$, the set

$$\mathcal{M}_j = \big\{u \in \mathcal{U} \,|\, \nabla\lambda_j(u)\cdot r_j(u) = 0\big\}$$

is a smooth affine manifold with dimension $N-1$ containing the point u_*. For simplicity in the presentation we do not include linearly degenerate fields.

We denote by $\mu_j(u)$ a scalar-valued function satisfying $\nabla\mu_j \cdot r_j \equiv 1$. When the j-field is genuinely nonlinear, one takes $\mu_j(u) = \lambda_j(u)$. The function μ_j will be used to parameterize the wave curves. We assume that μ_j can be chosen such that

$$\mu_j(u) = 0 \qquad \text{iff} \quad \nabla\lambda_j(u)\cdot r_j(u) = 0,$$

and either

$$\text{Case A: } \mu_j(u) \quad \text{and} \quad \nabla\lambda_j(u)\cdot r_j(u) \quad \text{have the same sign,} \tag{2.10a}$$

or

$$\text{Case B: } \mu_j(u) \quad \text{and} \quad \nabla\lambda_j(u)\cdot r_j(u) \quad \text{have the opposite sign.} \tag{2.10b}$$

In particular $\nabla\lambda_j \cdot r_j$ changes sign across $\mathcal{M}_j$. In Case A, $\mu_j(u) = 0$ is associated with a minimum of the wave speed, while in Case B it is associated with a maximum. In the scalar case, (2.10a) means that there is a state u_* such that the function f is strictly concave for $u < u_*$ and strictly convex for $u > u_*$. Typical examples are $f(u) = u^3$ in the case (2.10a) and $f(u) = -u^3$ in the case (2.10b); in both cases one can choose $\mu(u) = u$. As we will see, the cases (2.10a) and (2.10b) lead to wave curves with different properties.

The Riemann problem, (2.1) with initial data

$$u(x,0) = \begin{cases} u_l & \text{for } x < 0, \\ u_r & \text{for } x > 0, \end{cases} \tag{2.11}$$

and u_r and u_l fixed in $\mathcal{U}$, plays an important role in the theory of hyperbolic conservation laws. Since the problem is invariant under the transformation $(x,t) \to (\beta\, x, \beta\, t)$ (with $\beta > 0$), it is natural to search for self-similar solutions depending only on x/t. We now define the one-parameter families of shock and rarefaction waves to be used as building blocks in the resolution of the Riemann problem.

Given a state $u_0 \in \mathcal{U}$ and $j = 1, 2, \dots, N$, let $\mathcal{O}_j(u_0) = \big\{v_j(\epsilon_j; u_0) \in \mathcal{U}\big\}$ be the integral curve of the vector field r_j issued from u_0, so that

$$\frac{dv_j}{d\epsilon_j}(\epsilon_j; u_0) = r_j\big(v_j(\epsilon_j; u_0)\big), \qquad v_j(\epsilon_{j,0}; u_0) = u_0. \tag{2.12}$$

Note that $r_j(u_0)$ is the tangent vector of the curve $\mathcal{O}_j(u_0)$ at the point u_0. Using the normalization of the function μ_j, one checks that

$$\mu_j\big(v_j(\epsilon_j; u_0)\big) = \epsilon_j;$$

therefore there should be no confusion in using the notation $mu_j = \epsilon_j$. In other words, μ_j is viewed as both a function of u and as a parameter along the wave curves.

We also consider the Hugoniot locus

$$\mathcal{H}_j(u_0) := \big\{ w \,|\, -s\,\big(w - u_0\big) + f(w) - f(u_0) = 0 \big\}. \tag{2.13}$$

The Rankine–Hugoniot relation is equivalent to saying that there exists an index j and a scalar-valued coefficient $\alpha(u_0, w)$ such that

$$w - u_0 = \alpha(u_0, w)\, \bar{r}_j(u_0, w), \qquad s = \bar{\lambda}_j(u_0, w). \tag{2.14}$$

By the implicit function theorem, the Hugoniot set decomposes (locally near u_0, at least) into N Hugoniot curves $\mathcal{H}_j(u_0) = \big\{ w_j(\mu_j; u_0) \in \mathcal{U} \big\}$, passing through u_0 and having the tangent vector $r_j(u_0)$ at u_0. Since $\nabla \mu_j \cdot r_j > 0$, the coefficient $\alpha(u_0, w_j)$ in (2.14) has the same sign as that of $\mu_j(w_j) - \mu_j(u_0)$. Along the j-curve, the shock speed satisfies

$$\bar{\lambda}_j(u_0, w_j) = \lambda_j(u_0) + \frac{\mu_j}{2}\, \nabla \lambda_j(u_0) \cdot r_j(u_0) + O(\mu_j^2). \tag{2.15}$$

Taking a suitable subset $B(u_*, R')$ of $\mathcal{U} = B(u_*, R)$ if necessary, one can assume that the curves $\mathcal{O}_j(u_0)$ and $\mathcal{H}_j(u_0)$ extend up to the boundary of $\mathcal{U}$. Furthermore we assume that, for $j \in \mathbf{P}$, these curves are *transverse to the manifold* $\mathcal{M}_j$: each Hugoniot curve and each integral curve intersect the manifold at exactly one point. Observe that when R is small enough, it is sufficient to assume that the vector field r_j is transverse to the manifold $\mathcal{M}_j$. Our construction here applies, however, to the case that R is not necessarily small. The tranversality assumption implies that, for $j \in \mathbf{P}$, the wave speed $\mu_j \to \lambda_j\big(v_j(\mu_j; u_0)\big)$ has exactly one critical point along each integral curve. It will be checked in Lemma 2.3 below that, for $j \in \mathbf{P}$, the shock speed $\mu_j \to \bar{\lambda}_j(u_0, w_j(\mu_j; u_0))$ also admits (at most) one critical point along the Hugoniot curve.

Finally we introduce another assumption about the Hugoniot curve, for all $w_j(\mu_j; u_0)$ with $\mu_j \neq \mu_j(u_0)$,

$$l_j(w_j) \cdot \frac{dw_j}{d\mu_j} \,>\, 0, \tag{2.16i}$$

$$\big(\mu_j - \mu_j(u_0)\big)\, l_j(w_j) \cdot (w_j - u_0) \,>\, 0. \tag{2.16ii}$$

Both conditions in (2.16) trivially hold for weak shocks, since $l_j(u_0) \cdot r_j(u_0) = 1$.

Discontinuous solutions being not unique in general, it is customary to select the "admissible" weak solutions via an entropy criterion acting on discontinuities. From physical, mathematical, and numerical standpoints, it is desirable that an admissible solution to the Riemann problem exist, be unique, and depend continuously upon its initial states in a certain topology. In the classical approach, a wave curve $\mathcal{W}_j(u)$ is indeed defined by piecing together (admissible) parts of the above curves. The Lax shock inequalities [34, 35] are fundamental for stability and are used for weak shocks in the neighborhood of a point of genuine nonlinearity. A j-shock connecting u_0 to u_1 with speed $\lambda_j(u_0, u_1)$ is admissible in the sense of Lax iff

$$\lambda_j(u_0) \geq \bar{\lambda}_j(u_0, u_1) \geq \lambda_j(u_1). \tag{2.17}$$

Note that the inequalities $\lambda_{j-1}(u_0) < \bar{\lambda}_j(u_0, u_1) < \lambda_{j+1}(u_1)$ are obtained as a direct consequence of (2.9). When the characteristic fields are genuinely nonlinear, applying the Lax criterion leads to uniquely defined wave curves and to a unique solution for the Riemann problem. Each wave curve contains two distinct parts, half of the Hugoniot curve and half of the integral curve.

When one or more characteristic fields are not genuinely nonlinear, Liu proposed that, along the Hugoniot curve $\mathcal{H}_j(u_0)$, the following criterion holds:

$$\bar{\lambda}_j(u_0, w_j(\mu_j; u_0)) \ge \bar{\lambda}_j(u_0, u_1) \tag{2.18}$$

for all μ_j between $\mu_j(u_0)$ and $\mu_j(u_1)$; in other words, the shock speed for μ_j in the above range achieves its *minimum* at the point u_1. Liu [42] constructed a unique wave curve based on the condition (2.18). The wave curves may be composed of more than two pieces, and the Riemann solution contains composite waves mixing shocks and rarefactions.

It is known that (2.8), (2.17), and (2.18) are equivalent for shocks of weak amplitude and genuinely nonlinear fields. This is not true for systems having nongenuinely nonlinear fields. In the present paper we attempt to construct a wave curve based on (2.5) of the wave curves of Liu. However, instead of one-parameter wave curves we arrive here to two-parameter sets, which we call "wave sets." In this construction it is important to distinguish several types of discontinuities.

An arbitrary j-shock connecting u_0 to u_1 can be either *a Lax shock*, in which case (2.17) holds, *an undercompressive shock* satisfying either

$$\bar{\lambda}_j(u_0, u_1) \le \min\big(\lambda_j(u_0), \lambda_j(u_1)\big) \text{ or} \tag{2.19}$$

$$\bar{\lambda}_j(u_0, u_1) \ge \max\big(\lambda_j(u_0), \lambda_j(u_1)\big), \tag{2.20}$$

or *a rarefaction shock*:

$$\lambda_j(u_0) < \bar{\lambda}_j(u_0, u_1) < \lambda_j(u_1). \tag{2.21}$$

The properties of the wave speeds and shock speeds are described in Lemmas 2.3 and 2.4. (See Figure 2.1 for a graphical representation.) The entropy dissipation is dealt with in Lemma 2.5.

LEMMA 2.3. *Let u_0 be given with $\mu_j(u_0) > 0$ and consider the Hugoniot curve $\mathcal{H}_j(u_0)$ for $= j = 1, 2, \dots, N$. Suppose that* (2.10*a*) *(resp.,* (2.10*b*)*) holds. Then the wave speed $\mu_j \to g(\mu_j; u_0) := \lambda_j(w_j(\mu_j; u_0))$ is decreasing (resp., increasing) for $\mu_j < 0$ and increasing (resp., decreasing) for $\mu_j > 0$ and achieves its minimum (resp., maximum) at $\mu_j = 0$.*

There exists $\mu_j^\star(u_0) \le 0$ such that the shock speed $\mu_j \to h(\mu_j; u_0) := \bar{\lambda}_j(u_0, w_j(\mu_j; u_0))$ is decreasing (resp., increasing) for $\mu_j < \mu_j^\star(u_0)$ and increasing (resp., decreasing) for $\mu_j > \mu_j^\star(u_0)$ and achieves its minimum (resp., maximum) at $\mu_j^\star(u_0)$.

The wave speed and the shock speed coincide at the critical value of the shock speed:

$$g(\mu_j^\star(u_0); u_0) = h(\mu_j^\star(u_0); u_0). \tag{2.22}$$

Moreover we have in case (2.10*a*)

$$\begin{aligned} &h(\mu_j; u_0) - g(\mu_j; u_0) > 0 \qquad \text{for } \mu_j \in \big(\mu_j^\star(u_0), \mu_j(u_0)\big), \\ &h(\mu_j; u_0) - g(\mu_j; u_0) < 0 \qquad \text{for } \mu_j < \mu_j^\star(u_0) \text{ or } \mu_j > \mu_j(u_0), \end{aligned} \tag{2.23a}$$

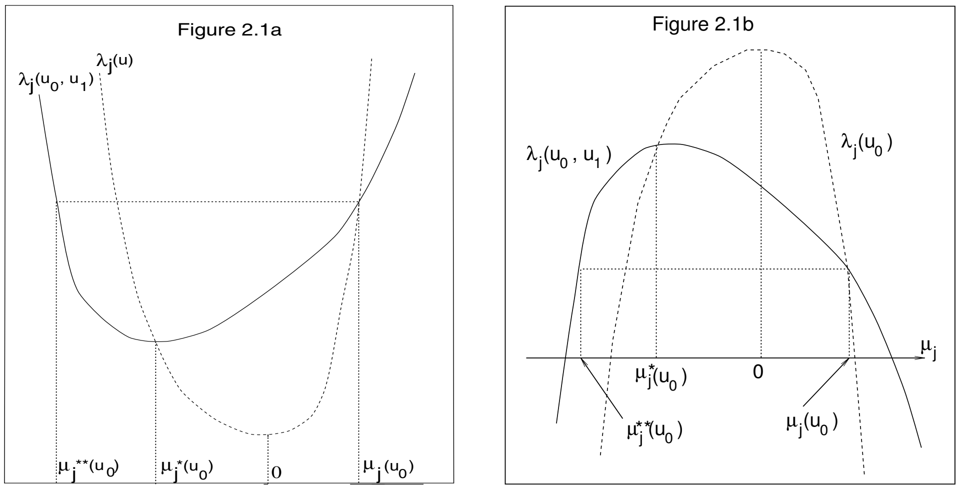


FIG. 2.1. *Wave speed and shock speed for* (a) *the case* (2.10a), (b) *the case* (2.10b).

and in the case (2.10b) *we have*

$$h(\mu_j;u_0) - g(\mu_j;u_0) < 0 \qquad \text{for } \mu_j \in \big(\mu_j^\star(u_0), \mu_j(u_0)\big),$$
$$h(\mu_j;u_0) - g(\mu_j;u_0) > 0 \qquad \text{for } \mu_j < \mu_j^\star(u_0) \text{ or } \mu_j > \mu_j(u_0). \tag{2.23b}$$

When $\mu_j(u_0) = 0$, the same properties hold with $\mu_j^\star(u_0) = 0$.

Lemma 2.3 includes, as a special case, the situation that the point $w_j(\mu_j^\star(u_0);u_0)$ belongs to the boundary of $\mathcal{U}$, in which case $\big\{\mu_j < \mu_j^\star(u_0)\big\}$ is empty. We denote by $\mu_j^{\star\star}(u_0)$, with $\mu_j^{\star\star}(u_0) < \mu_j^\star(u_0)$, the point of the Hugoniot curve such that

$$h(\mu_j^{\star\star}(u_0);u_0) = h(\mu_j(u_0);u_0) \tag{2.24}$$

when such a point exists. In the following, we tacitly assume that both points, $\mu_j^\star(u_0)$ and $\mu_j^{\star\star}(u_0)$, exist and belong to the interior of $\mathcal{U}$, the discussion below being much simpler in other cases. Lemma 2.3 is due to Liu [42] and, for completeness, a proof is given in the appendix.

LEMMA 2.4. *Let u_0 be given with $\mu_j(u_0) \geq 0$ and consider the Hugoniot curve $\mathcal{H}_j(u_0)$.*

(1) *Suppose that* (2.10a) *holds. A shock connecting u_0 to $u_1 = w_j(\mu_j(u_1);u_0)$ is*

$$\text{a rarefaction shock if } \mu_j(u_1) > \mu_j(u_0) \text{ or } \mu_j(u_1) < \mu_j^{\star\star}(u_0);$$
$$\text{a Lax shock if } \mu_j(u_1) \in \big[\mu_j^\star(u_0), \mu_j(u_0)\big]; \tag{2.25a}$$
$$\text{an undercompressive shock if } \mu_j(u_1) \in \big[\mu_j^{\star\star}(u_0), \mu_j^\star(u_0)\big).$$

In the second case the shock also satisfies the (stronger) Liu criterion.

(2) *Suppose that* (2.10b) *holds. A shock connecting u_0 to $u_1 = w_j(\mu_j(u_1);u_0)$ is*

$$\text{a Lax shock if } \mu_j(u_1) \geq \mu_j(u_0) \text{ or } \mu_j(u_1) \leq \mu_j^{\star\star}(u_0);$$
$$\text{a rarefaction shock if } \mu_j(u_1) \in \big(\mu_j^\star(u_0), \mu_j(u_0)\big); \tag{2.25b}$$
$$\text{an undercompressive shock if } \mu_j(u_1) \in \big(\mu_j^{\star\star}(u_0), \mu_j^\star(u_0)\big].$$

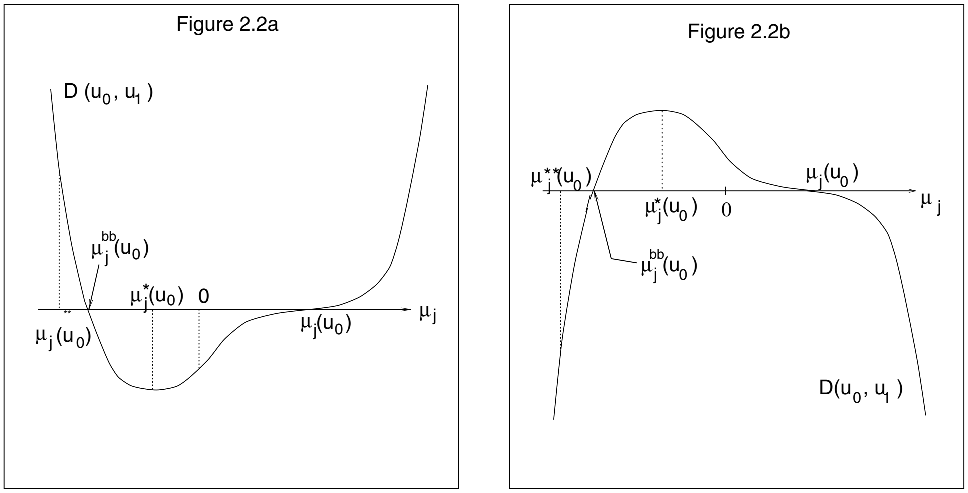


FIG. 2.2. *Entropy dissipation for* (a) *the case* (2.10a), (b) *the case* (2.10b).

In the first case the shock also satisfies the (stronger) Liu criterion.

LEMMA 2.5. *Let u_0 be given with $\mu_j(u_0) \geq 0$ and consider the Hugoniot curve $\mathcal{H}_j(u_0)$. Suppose that* (2.10a) *(resp.,* (2.10b)*) holds.*

(1) *The entropy dissipation $\mu_j \to D(u_0, w_j(\mu_j; u_0))$ vanishes at $\mu_j(u_0)$ and at a point $\mu_j^{\flat\flat}(u_0)$ in the interval $\big(\mu_j^{\star\star}(u_0), \mu_j^\star(u_0)\big)$. The entropy dissipation is decreasing (resp., increasing) for $\mu_j < \mu_j^\star(u_0)$, increasing (resp., decreasing) for $\mu_j > \mu_j^\star(u_0)$, and achieves a negative maximum value (resp., a positive maximum value) at the critical point of the wave speed, that is, $\mu_j^\star(u_0)$.*

(2) *A shock satisfying* (2.8) *cannot be a rarefaction shock. As a corollary, a nonclassical shock is undercompressive and satisfies $\mu_j \in \big(\mu_j^{\flat\flat}(u_0), \mu_j^\star(u_0)\big)$ (resp., $\mu_j \in \big(\mu_j^{\star\star}(u_0), \mu_j^{\flat\flat}(u_0)\big)$).*

(3) *Any shock satisfying the Liu criterion* (2.18) *also satisfies the entropy inequality* (2.8).

For u_l and u_r given in $\mathcal{U}$, the Riemann problem (2.1), (2.11) admits up to a P-parameter family of solutions containing N separated wave fans, each of them being composed of (at most) two waves. Specifically we obtain the following description of the classical and nonclassical waves.

Consider a j-wave fan with left-hand state u_0 and right-hand state u with $\mu_j(u_0) \geq 0$. For $j \notin \mathbf{P}$, the wave fan is either a rarefaction wave if $\mu_j(u) > \mu_j(u_0)$, or a classical shock if $\mu_j(u) < \mu_j(u_0)$. For $j \in \mathbf{P}$, we have the following.

Case A. Assume that (2.10a) holds and $j \in \mathbf{P}$. Assume first that $\mu_j(u_0) > 0$. The j-wave fan using only classical waves contains

(1) either a rarefaction from u_0 to $u \in \mathcal{O}_j(u_0)$ if $\mu_j(u) > \mu_j(u_0)$,

(2) a classical shock from u_0 to $u \in \mathcal{H}_j(u_0)$ if $\mu_j(u) \in \big(\mu_j^\star(u_0), \mu_j(u_0)\big)$,

(3) or a classical shock from u_0 to $u^\star := w_j(\mu^\star(u_0); u_0\big)$ followed by an attached rarefaction connecting to $u \in \mathcal{O}_j(u^\star)$ if $\mu_j(u) < \mu_j^\star(u_0)$.

This completes the description of the classical wave curve $\mathcal{W}_j^c(u_0)$ for Case A.

THEOREM 2.6A. *The j-wave fan may also contain a nonclassical j-shock connecting u_0 to* any state *$u^\flat \in \mathcal{H}_j(u_0)$ with $\mu_j(u^\flat) \in \big(\mu_j^{\flat\flat}(u_0), \mu_j^\star(u_0)\big)$ followed by*

(1) *either a* nonattached *rarefaction connecting $u^\flat$ to $u \in \mathcal{O}_j(u^\flat)$ if $\mu_j(u) < \mu_j(u^\flat)$,*

(2) *or by a classical shock connecting* $u^\flat$ *to* $u \in \mathcal{H}_j(u^\flat)$ *if* $\mu_j(u) > \mu_j(u^\flat)$.

This defines a two-parameter family of u *that can be reached from* u_0 *by nonclassical solutions. For a given* $u^\flat$, *the classical shock with largest strength and connecting* $u^\flat$ *to some* $u = u^\sharp \in \mathcal{H}_j(u^\flat)$ *is characterized by the condition* $\bar\lambda_j(u^\flat, u^\sharp) = \bar\lambda_j(u_0, u^\flat)$ *and, in that situation, one also has* $u^\sharp \in \mathcal{H}_j(u_0)$. *In particular the nonclassical shock with largest possible strength connects the point* $u^{\flat\flat} := w_j(\mu_j^{\flat\flat}(u_0); u_0)$ *to the point* $u^{\sharp\sharp} := w_j(\mu^{\sharp\sharp}(u_0); u^{\flat\flat})$, *where* $\mu^{\sharp\sharp}(u_0)$ *is defined by* $u^{\sharp\sharp} \in H_j(u_0)$. *Moreover one has*

$$\mu_j^{\star\star}(u_0) \le \mu_j^{\flat\flat}(u_0) \le \mu_j^{\flat}(u_0) \le \mu_j^{\star}(u_0) \le \mu_j^{\sharp}(u_0) \le \mu_j^{\sharp\sharp}(u_0) \le \mu_j(u_0). \tag{2.26}$$

In the special case that $\mu_j(u_0) = 0$, the j-wave curve is the j-integral curve issuing from u_0.

Case B. Assume that (2.10b) holds and $j \in \mathbf{P}$. Assume first that $\mu_j(u_0) > 0$. The j-wave fan using only classical waves contains

(1) either a classical shock connecting u_0 to $u \in \mathcal{H}_j(u_0)$ if either $\mu_j(u) \ge \mu_j(u_0)$ or $\mu_j(u) \le \mu_j^{\star\star}(u_0)$,

(2) a rarefaction connecting u_0 to $u \in \mathcal{O}_j(u_0)$ if $\mu_j(u) \in \big[0, \mu_j(u_0)\big]$,

(3) or a rarefaction wave connecting u_0 to a point u_1, followed by an attached classical shock connecting to $u \in \mathcal{H}_j(u_1)$ with $\mu_j(u) = \mu_j^{\star\star}(u_1)$, if $\mu_j(u) \in \big(\mu_j^{\star\star}(u_0), 0\big)$. (In this case the set of u does not describe a rarefaction or shock curve.)

This completes the description of the classical wave curve $\mathcal{W}_j^c(u_0)$.

THEOREM 2.6B. *The* j*-wave fan may also contain*

(1) *either a rarefaction to* $u \in \mathcal{O}_j(u_0)$ *if* $\mu_j(u) \in \big(0, \mu_j(u_0)\big)$, *possibly followed by a* nonattached *nonclassical shock connecting* u_1 *to* u, *if* $\mu_j(u) \in \big(\mu_j^{\star\star}(u_1), \mu_j^{\flat\flat}(u_1)\big)$ *(in this case the set of* u *does not describe a rarefaction or shock curve),*

(2) *or a classical shock to* $u_1 \in \mathcal{H}_j(u_0)$ *with* $\mu_j(u_1) > \mu_j(u_0)$, *followed by a nonclassical shock connecting to* $u \in \mathcal{H}_j(u_1)$, *if* $\mu_j(u) \in \big(\mu_j^{\star\star}(u_1), \mu_j^{\flat\flat}(u_1)\big)$.

This defines a two-parameter family of u *that can be reached from* u_0 *by nonclassical solutions.*

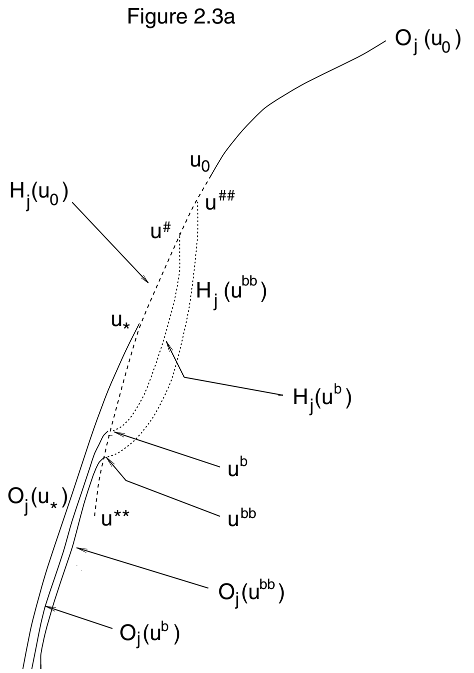
Assume finally that $\mu_j(u_0) = 0$. Then the j-wave curve is the j-Hugoniot curve issuing from u_0 and correspond to classical shocks.

Based on these results, we introduce the following terminology. Given u_0, the set of all states that can be reached using only j-waves will be called the j*-wave set* issuing from u_0 and be denoted by $\mathcal{S}_j(u_0)$ by analogy with the notion of j-wave curve known for classical solutions. We shall call a curve in the wave set a *composite curve* when it is not a part of a rarefaction or shock curve. The wave set in both cases (2.10a) and (2.10b) is represented in Figures 2.3(a) and 2.3(b), respectively. The case that $\mu_j(u_0) < 0$ is analogous and is omitted. We now give a proof of Lemmas 2.4 and 2.5 and Theorem 2.6.

REMARK 2.7. (1) Our analysis shows that, under the assumptions made in this section, the Lax inequalities and the Liu criterion are *equivalent* (Lemma 2.4), which, at first, may appear surprising. The Lax inequalities are sufficient to select a unique solution for shocks with small amplitude near a point where $\nabla\lambda_j \cdot r_j$ vanishes. The Liu criterion is necessary for shocks of moderate amplitude when the product $\nabla\lambda_i \cdot r_i$ changes sign several times along the Hugoniot curve.

(2) When the system (2.1) has a sufficiently large family of entropies (e.g., when $N \le 2$), the formulas (2.28)–(2.29) derived below may be used to establish the converse of item (3) of Lemma 2.5, i.e., limits of regularizations compatible with all entropies (such as (2.2) with $D_\epsilon = \epsilon\, \partial_x u_\epsilon$), necessarily satisfy the Liu criterion.

(3) It may be of interest to search for the weakest constraint on undercompressive



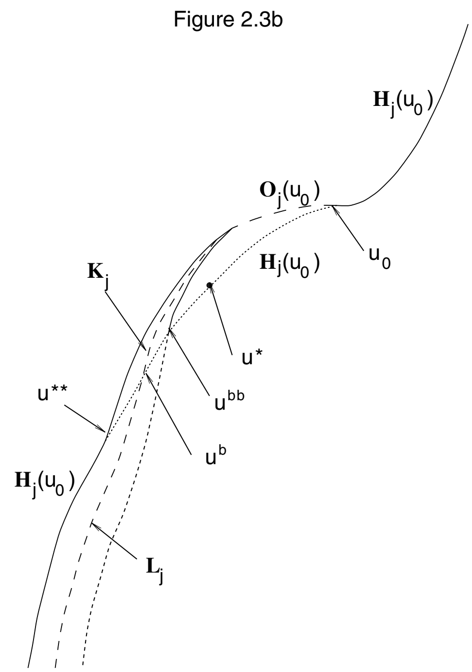


FIG. 2.3. *Wave set* $\mathcal{S}_j(u_0)$ *issuing from* u_0 *for* (a) *the case* (2.10a), (b) *the case* (2.10b).

shocks that can result from imposing one entropy inequality like (2.8). We shall say that a subset $\mathcal{W}_j^{max}(u_0)$ of $\mathcal{U}$ is a *maximal* j*-wave set* for the system (2.1) if it contains all the j-wave sets for arbitrary entropies. For instance a maximal wave set for the case (2.10a) is obtained by taking $\mu_j^{\flat\flat}(u_0) = \mu_j^{\star\star}(u_0)$ in Theorem 2.6; this follows readily from the formula (2.28)–(2.29). In the scalar case with $N = 1$ and $f(u) = u^3$, one has $\mu_j(u_0) = u_0$ and $\mu_j^{\star\star}(u_0) = -2\,u_0$. The scalar case is degenerated and $\mathcal{W}^c(u_0) = \mathcal{W}^{nc}(u_0) = \mathcal{W}^{max}(u_0) = \mathbb{R}$; the interval $[-2\,u_0, u_0]$ is the maximal interval of states that can be reached from u_0 by using a classical or nonclassical shock. ◻

Proof of Lemma 2.4. Consider for instance the case (2.10a), the case (2.10b) being similar. Lemma 2.3 states that the function $\mu_j \to \bar\lambda_j(u_0, w_j(\mu_j; u_0)) - \bar\lambda_j(w_j(\mu_j; u_0))$ is positive for $\mu_j > \mu_j^\star(u_0)$ and negative for $\mu_j < \mu_j^\star(u_0)$. On the other hand the function $\mu_j \to \bar\lambda_j(u_0, w_j(\mu_j; u_0)) - \lambda_j(u_0)$ is positive for $\mu_j < \mu_j^{\star\star}(u_0)$ or $\mu_j > \mu_j(u_0)$ and negative for $\mu_j \in \big(\mu_j^{\star\star}(u_0), \mu_j(u_0)\big)$. The classification follows easily from these two properties. ◻

Proof of Lemma 2.5. Using the compatibility condition on the entropy pair, i.e., $\nabla F^T = \nabla U^T Df$, and the Rankine–Hugoniot relation (2.13), the entropy dissipation for a shock connecting u_0 to $w_j(\mu_j; u_0)$ is found to be

(2.27)
$$
\begin{aligned}
&D(u_0, w_j(\mu_j; u_0))\\
&= \int_{\mu_j(u_0)}^{\mu_j} \nabla U(w_j(\zeta_j)) \left\{\bar\lambda_j(u_0, w_j(\mu_j)) - Df(w_j(\zeta_j))\right\} \frac{dw_j}{d\zeta_j}(\zeta_j)\, d\zeta_j,\\
&= \int_{\mu_j(u_0)}^{\mu_j} \frac{dw_j}{d\zeta_j}(\zeta_j) \cdot \nabla^2 U(w_j(\zeta_j)) \left\{\bar\lambda_j(u_0, w_j(\zeta_j)) \big(w_j(\zeta_j) - u_0\big) - f(w_j(\zeta_j)) + f(u_0)\right\} d\zeta_j.
\end{aligned}
$$

Using once more the Rankine–Hugoniot relation, we get

$$D(u_0, w_j) = \int_{\mu_j(u_0)}^{\mu_j} \big\{ \bar\lambda_j(u_0, w_j(\mu_j)) - \bar\lambda_j(u_0, w_j(\zeta_j)) \big\}\, m_j(\zeta_j)\, d\zeta_j, \tag{2.28}$$

where

$$m_j(\zeta_j) := \frac{dw_j}{d\zeta_j}(\zeta_j) \cdot \nabla^2 U(w_j(\zeta_j))\, \big(w_j(\zeta_j) - u_0\big) \tag{2.29}$$

has the same sign as $\mu_j - \mu_j(u_0)$. The system (2.1) being strictly hyperbolic, it can be checked that

$$\frac{dw_j}{d\mu_j} \cdot \nabla^2 U(w_j) \,=\, l_j(w_j),$$

which, combined with (2.16ii), shows that $m_j(\zeta_j) > 0$ for $\zeta_j \neq \mu_j(u_0)$.

The occurrence of nonclassical shocks depends on the sign of the entropy dissipation. The integrand in (2.28) has the same sign as $\bar\lambda_j(u_0, w_j(\mu_j)) - \bar\lambda_j(u_0, w_j(\zeta_j))$, which is nonpositive when the Liu entropy criterion (2.18) holds. It follows that the entropy dissipation is negative as long as the Liu criterion holds. This proves item (3) of Lemma 2.5.

When, instead, the shock satisfies the inequalities (2.21), we have

$$\bar\lambda_j(u_0, w_j(\mu_j)) - \bar\lambda_j(u_0, w_j(\zeta_j)) \geq 0. \tag{2.30}$$

This indeed is an easy consequence of the facts that $\mu_j \to \bar\lambda_j(u_0, w_j(\mu_j))$ is a monotone function (see Lemma 2.3) and that $\bar\lambda_j(u_0, u_0) = \lambda_j(u_0) \leq \bar\lambda_j(u_0, w_j(\mu_j))$. Combining (2.28) and (2.30) shows that the entropy dissipation is negative for rarefaction shocks. This proves item (2) of Lemma 2.5.

Finally we can establish item (1) by differentiating the formula (2.28) with respect to μ_j:

$$\frac{\partial}{\partial \mu_j} D(u_0, w_j) = \int_{\mu_j(u_0)}^{\mu_j} \frac{\partial}{\partial \mu_j} \bar\lambda_j(u_0, w_j)\, m_j(\zeta_j)\, d\zeta_j.$$

This yields a relation between the derivative of the entropy dissipation and that of the shock speed:

$$\frac{\partial}{\partial \mu_j} D(u_0, w_j) = b(w_j)\, \frac{\partial}{\partial \mu_j} \lambda_j(u_0, w_j), \qquad b(w_j) := \int_{\mu_j(u_0)}^{\mu_j} m_j(\zeta_j)\, d\zeta_j, \tag{2.31}$$

with $C_1\, |w_j - u_0|^2 \leq b(w_j) \leq C_2\, |w_j - u_0|^2$ for some positive constants C_1 and C_2. Note that the dissipation has a critical point either when the shock speed has a critical point or at the point u_0.

From the properties of the shock speed in Lemma 2.3, it follows therefore that $D(u_0, w_j)$ is decreasing for $\mu_j < \mu_j^\star(u_0)$ and increasing for $\mu_j > \mu_j^\star(u_0)$. From its definition, it is clear that $D(u_0, w_j)$ vanishes at $\mu_j(u_0)$. Moreover, we checked that it is positive for $\mu_j < \mu_j^{\star\star}(u_0)$. Therefore there exists a unique point, say, $\mu_j^{\flat\flat}(u_0)$, in the interval $\big(\mu_j^{\star\star}(u_0), \mu_j^\star(u_0)\big)$ where the dissipation vanishes. This completes the proof of Lemma 2.5. ∎

Proof of Theorem 2.6. We construct the wave set $\mathcal{S}_j(u_0)$ for $u_0 \in \mathcal{U}$ and $j \in \mathbf{P}$. The construction for $j \notin \mathbf{P}$ is classical and $\mathcal{S}_j(u_0)$ is the classical wave curve $\mathcal{W}_j(u_0)$.

Case A. For $u_0 \in \mathcal{M}_j$, either of the conditions (2.8) or (2.18) shows that the wave set $\mathcal{W}_j^{nc}(u_0)$ coincides locally with the integral curve $\mathcal{O}_j(u_0)$. This is because the wave speed is increasing when moving away from u_0 in either direction. The construction is complete for $u_0 \in \mathcal{M}_j$.

We now consider a point u_0 away from the manifold. For definiteness we assume that $\mu_j(u_0) > 0$; the other case could be treated similarly. The construction of the wave curve will use the values $\mu_j^{\star\star}(u_0) < \mu_j^{\star}(u_0) \leq \mu_j(u_0)$ introduced in Lemma 2.3.

For $\mu_j > \mu_j(u_0)$, the state u_0 can be connected to any point on $\mathcal{O}_j(u_0)$ since the wave speed λ_j is increasing for μ_j increasing. Therefore the wave curve $\mathcal{W}_j(u_0)$ coincides with the rarefaction curve $\mathcal{O}_j(u_0)$ for $\mu_j \geq \mu_j(u_0)$.

For μ_j decreasing from $\mu_j(u_0)$, the shock speed is decreasing as long as μ_j remains larger than the critical value $\mu_j^{\star}(u_0)$. Therefore all the points in the Hugoniot curve $\mathcal{H}_j(u_0)$ with $\mu_j \in [\mu_j^{\star}(u_0), \mu_j(u_0)]$ can be reached from u_0 by a classical shock satisfying the Liu criterion. According to Lemma 2.5, the entropy dissipation remains negative in the whole range $\mu_j \in [\mu_j^{\flat\flat}(u_0), \mu_j(u_0)]$. Thus the points of the Hugoniot curve $\mathcal{H}_j(u_0)$ with $\mu_j \in [\mu_j^{\flat\flat}(u_0), \mu_j^{\star}(u_0)]$ can also be reached from u_0 but, now, with a nonclassical shock.

These are the only admissible solutions with a single j-wave issuing from u_0.

Consider now an admissible one-wave solution joining u_0 to u_1. If $\mu_j(u_1) > \mu_j^{\star}(u_0)$, then no further j-wave can be constructed from u_1. The state $u_1^{\star}$ with $\mu_j(u_1) = \mu_j^{\star}(u_0)$ can be connected to any point u_2 in the rarefaction curve $\mathcal{O}_j(u_1)$ with $\mu_j(u_2) \leq \mu_j^{\star}(u_0)$. This covers the whole range of values μ_j and corresponds to the classical wave curve.

We now describe all nonclassical solutions with two j-waves. Consider an admissible one-wave solution joining u_0 to $u^\flat$ with $\mu_j(u^\flat) \in \big[\mu_j^{\flat\flat}(u_0), \mu_j^{\star}(u_0)\big)$. According to Lemma 2.3, the wave speed is increasing with μ_j decreasing from $\mu_j(u^\flat)$, so $u^\flat$ can be connected to any point u_2 in the rarefaction curve $\mathcal{O}_j(u^\flat)$ with $\mu_j(u_2) \leq \mu_j(u^\flat)$. Observe that the nonclassical shock is not attached to the rarefaction fan, i.e.,

$$\bar{\lambda}_j(u_0, u^\flat) < \lambda_j(u^\flat). \tag{2.32}$$

This describes all the solutions containing a nonclassical shock followed by a rarefaction; no further j-wave may follow the rarefaction.

Consider again an admissible one-wave solution joining u_0 to $u^\flat$ with $\mu_j(u^\flat) \in \big[\mu_j^{\flat\flat}(u_0), \mu_j^{\star}(u_0)\big)$. By (2.32), the shocks with small strength issuing from $u^\flat$ have a larger speed than that of the nonclassical shock, i.e., $\bar{\lambda}_j(u^\flat, u_2) \approx \lambda_j(u^\flat) > \bar{\lambda}_j(u_0, u^\flat)$ for all states u_2 close to $u^\flat$. Hence the speeds have the proper ordering and $u^\flat$ may be connected to any $u_2 \in \mathcal{H}_j(u^\flat)$, at least in the small. Such a shock is also admissible (according to the Liu criterion) since the wave speed is decreasing when μ_j increases (Lemma 2.3.).

This construction can be continued, for $u^\flat$ fixed, until u_2 violates either of the two conditions

$$\bar{\lambda}_j(u^\flat, u_2) > \bar{\lambda}_j(u_0, u^\flat) \text{ or} \tag{2.33}$$

$$D(u^\flat, u_2) \leq 0. \tag{2.34}$$

Actually, as $\mu_j(u_2)$ increases from $\mu_j(u^\flat)$, one reaches a maximum value $\mu_j^\sharp$, in which equality holds in (2.33), while the shock is still classical and therefore (2.34) still holds.

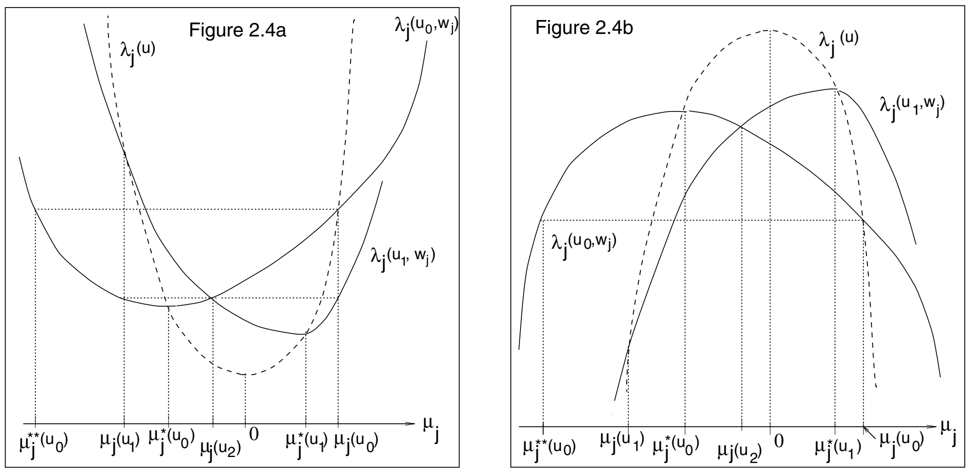


FIG. 2.4. *Graphs of the two shock speeds for* (a) *the case* (2.10a), (b) *the case* (2.10b).

To check the latter, consider the graphs of the two functions $h(\mu_j) := \bar\lambda_j(u_0, w_j(\mu_j; u_0))$ and $k(\mu_j) := \bar\lambda_j(u^\flat, w_j(\mu_j; u^\flat))$. See Figure 2.4(a). By symmetry of the Rankine–Hugoniot relation, one has $\bar\lambda_j(u^\flat, u_0) = \bar\lambda_j(u_0, u^\flat)$, so

$$s := h(\mu_j(u^\flat)) = k(\mu_j(u_0)). \tag{2.35}$$

In view of their monotonicity properties, the two graphs must intersect at exactly one point $\mu_j^\sharp$ in the interval $\big(\mu_j(u^\flat), \mu_j(u_0)\big)$. We define $u_2^\sharp$ by the conditions $\mu_j(u_2^\sharp) = \mu_j^\sharp$ and $u_2^\sharp \in \mathcal{H}_j(u^f lat)$.

We claim that, actually,

$$h(\mu_j^\sharp) = k(\mu_j^\sharp) = s \qquad \text{and} \qquad u_2^\sharp \in \mathcal{H}_j(u_0). \tag{2.36}$$

Namely, from the Rankine–Hugoniot relations

$$-s\big(u^\flat - u_0\big) + f(u^\flat) - f(u_0) \,=\, 0 \qquad \text{and} \qquad -\,s\big(u_2^\sharp - u^\flat\big) + f(u_2^\sharp) - f(u^\flat) \,=\, 0,$$

we deduce that $-s\big(u_2^\sharp - u_0\big) + f(u_2^\sharp) - f(u_0) \,=\, 0$, which proves (2.36).

It follows (see Figure 2.4(a)) that (2.33) holds for all $\mu_j(u_2) < \mu_j(u_2^\sharp)$, and the equality holds in (2.33) at the critical value $u_2^\sharp$. Moreover, since $\mu_j(u_2^\sharp) < \mu_j^\star(u_0)$, the shock speed is decreasing on the interval $\big(\mu_j(u^\flat), \mu_j(u_2^\sharp)\big)$ and any shock from $u^\flat$ to u_2 (with $\mu_j(u_2) \le \mu_j(u_2^\sharp)$) satisfies the Liu criterion.

We have the inequalities $\mu_j(u^\flat) < \mu_j^\star(u_0) < \mu_j(u_2^\sharp) < \mu_j(u_0)$. As $\mu_j(u^\flat)$ increases, $\mu_j(u_2^\sharp)$ decreases and eventually both quantities approach the limiting value $\mu_j^\star(u_0)$. As $\mu_j(u^\flat)$ decreases, $\mu_j(u_2^\sharp)$ increases and eventually $\mu_j(u^\flat)$ approaches the limiting value $\mu_j^{\flat\flat}(u_0)$, while $\mu_j(u_2^\sharp)$ approaches some limiting value, say, $\mu_j^{\sharp\sharp}(u_0)$. It is tedious but straightforward to check from the properties of the wave speeds that no third wave can follow a two-wave fan. See Figure 2.2(a) for a representation of the wave set $\mathcal{S}_j(u_0)$.

Case B. For $u_0 \in \mathcal{M}_j$, it is not hard to see, using either of the conditions (2.8) or (2.18), that $\mathcal{W}_j(u_0)$ coincides locally with the Hugoniot curve $\mathcal{H}_j(u_0)$. This is because the wave speed is decreasing when moving away from u_0 in either direction. The construction is complete for $u_0 \in \mathcal{M}_j$.

Consider the case $\mu_j(u_0) > 0$. For $\mu_j > \mu_j(u_0)$, the state u_0 can be connected to any point on $\mathcal{H}_j(u_0)$ since the wave speed is decreasing for μ_j increasing. For $\mu_j < \mu_j(u_0)$, the wave speed is, locally, increasing for μ_j decreasing. So u_0 can be connected to a point on $\mathcal{O}_j(u_0)$ by a rarefaction. This remains possible until μ_j reaches the value 0. It is also possible to connect any point $u_1 \in \mathcal{O}_j(u_0)$ with $\mu_j(u_1) \in \big[0, \mu_j(u_0)\big]$ to a point $u_2 \in \mathcal{H}_j(u_1)$ provided

$$\bar\lambda_j(u_1, u_2) = \lambda_j(u_1). \tag{2.37}$$

This construction covers the range $\mu_j \in \big[\mu_j^{\star\star}(u_0), 0\big]$. It is also possible to connect u_0 directly to a point $u \in \mathcal{H}_j(u_0)$ with $\mu_j(u) \le \mu_j^{\star\star}(u_0)$, since the shock speed in this range satisfies the Liu criterion.

This completes the construction of the classical wave curve $\mathcal{W}_j^c(u_0)$.

We now describe all nonclassical solutions with two j-waves. Consider an admissible one-wave solution from u_0 to u_1. Suppose first $\mu_j(u) \in \big(0, \mu_j(u_0)\big)$ so that $u_1 \in \mathcal{O}_j(u_0)$. One can connect u_1 to $u_2 \in \mathcal{H}_j(u_0)$ by a shock provided both conditions

$$\bar\lambda_j(u_1, u_2) \ge \lambda(u_1), \tag{2.38}$$

$$D(u_1, u_2) \le 0 \tag{2.39}$$

hold. From the graph of the entropy dissipation, we know that (2.39) is equivalent to

$$\mu_j(u_2) \le \mu_j^{\flat\flat}(u_0).$$

In view of the graph of the shock speed, (2.38) reads

$$\mu_j^{\flat\flat}(u_1) \le \mu_j(u_2) \le \mu_j^{\star}(u_1).$$

Since we always have $\mu_j^{\flat\flat}(u_0) \in \big[\mu_j^{\star\star}(u_0), \mu_j^{\star}(u_0)\big]$, it follows that the admissible interval in the case under consideration is $\mu_j(u_2) \in \big[\mu_j^{\star\star}(u_0), \mu_j^{\flat\flat}(u_0)\big]$. Moreover such a shock is classical only when $\mu_j(u_2) \le \mu_j^{\star\star}(u_0)$, that is, only when $\mu_j(u_2) = \mu_j^{\star\star}(u_0)$.

Suppose now that $\mu_j(u) \ge \mu_j(u_0)$ so that $u_1 \in \mathcal{H}_j(u_0)$. One can connect u_1 to a point $u_2 \in \mathcal{H}_j(u_1)$ provided

$$\bar\lambda_j(u_1, u_2) \ge \bar\lambda_j(u_0, u_1) \tag{2.40}$$

and

$$D(u_1, u_2) \le 0. \tag{2.41}$$

The condition (2.41) is equivalent to saying $\mu_j(u_2) \le \mu_j^{\flat\flat}(u_1)$. As μ_j decreases from $\mu_j^{\flat\flat}(u_0)$, the speed $\bar\lambda_j(u_1, u_2)$ satisfies (2.40), decreases, and eventually reaches the value $\bar\lambda_j(u_0, u_1)$. Since $u_1 \in \mathcal{H}_j(u_0)$ and $u_2 \in \mathcal{H}(u_1)$, the same argument as in the case (2.10a) shows that for that value of μ_j, one has $u_2 \in \mathcal{H}_j(u_0)$. This completes the proof of Theorem 2.6. $\square$

2.3. Selection by kinetic relations. In view of Theorem 2.6, the wave set $\mathcal{S}_j(u_0)$ is a two-dimensional manifold when $j \in \mathbf{P}$. It is our objective now to select a nonclassical wave curve $\mathcal{W}_j^{nc}(u_0)$ in the wave set. Heuristically, it is sufficient to determine one free parameter needed for each nongenuinely nonlinear wave family. One may postulate that for each state u_0, there exists a *single* right state u_1 that can be reached by a nonclassical shock for any $j \in \mathbf{P}$. This is indeed what happens when defining nonclassical shocks as limits of diffusive-dispersive regularizations. We propose to select the admissible nonclassical shocks by considering their entropy dissipation and stipulate the knowledge of an additional jump-like relation on the nonclassical discontinuities. The derivation of such an additional relation for limits of diffusive-dispersive regularizations is discussed later in this subsection.

The following definition stipulates that the entropy dissipation

$$D(u_0, u_1) = -s\ (U(u_1) - U(u_0)) + F(u_1) - F(u_0)$$

of a nonclassical shock, with speed $s = \bar{\lambda}_j(u_0, u_1)$ and connecting u_0 to u_1, is a given "constitutive function" representing certain small-scale properties that have been neglected at the hyperbolic level of modeling. In the following we suppose, for the sake of definiteness, that the condition (2.10a) is satisfied. Dealing with the case (2.10b) requires some modification of the analysis in this subsection. (See also section 3 in which both cases arise.)

We denote by $BV \cap L^\infty$ the space of measurable and bounded functions that have bounded variation in space and time. This space is natural for systems of conservation laws. Functions in $BV \cap L^\infty$ admit traces in a measure theoretic sense [14], so that (2.42) below has a meaning almost everywhere with respect to the one-dimensional Hausdorff measure.

DEFINITION 2.8. *For each $j \in \mathbf{P}$, let $\phi_j : \mathcal{U} \to \mathbb{R}_-$ be a given function. A solution $u(x,t) \in BV \cap L^\infty$ to* (2.1), (2.8) *is called an admissible nonclassical entropy solution if it satisfies the entropy inequality and the entropy dissipation of any nonclassical j-shock in u ($j \in \mathbf{P}$), connecting u_0 to u_1, satisfies*

$$D(u_0, u_1) \;=\; \phi_j(u_0). \tag{2.42}$$

We refer to (2.42) as a *kinetic relation* and to ϕ_j as the *kinetic function* for the family j since they determine the propagation of the nonclassical shocks. The kinetic function could also be expressed as a function of the right state u_1 (which need not be equivalent to (2.42)) or—and this is physically more realistic—as a function of a variable "symmetric" in u_0 and u_1, such as the shock speed, or—for problems in fluid dynamics and material science—the mass flux across the discontinuity, etc. Here we shall focus attention on kinetic functions depending solely on the shock speed s, i.e.,

$$D(u_0, u_1) \;=\; \varphi(s), \tag{2.43}$$

where φ need be defined only on the union of intervals $\Lambda = \bigcup_{j \in \mathbf{P}} \left[\lambda_j^{\min}, \lambda_j^{\max}\right]$. For scalar conservation laws and under suitable monotonicity conditions, the kinetic function can always be expressed as a function of the shock speed. The same is true for the kinetics generated by diffusive-dispersive regularizations for the systems of two equations studied later in sections 3–5.

In many physical systems, the entropy dissipation is related to the mechanical energy and may be viewed as *a force driving the propagation* of the nonclassical shocks; it is natural to provide a one-to-one relationship between the propagation speed and the driving force. This standpoint was emphasized by Abeyaratne and Knowles [1] for propagating phase boundaries in solids undergoing phase transformations.

In the following we show that the kinetic relation selects a unique curve $\mathcal{W}_j^{nc}(u_0)$ corresponding to nonclassical solutions in the wave set $\mathcal{S}_j(u_0)$. Denote by $D_j^\star(u_0)$ the *maximal negative value* of the entropy dissipation $D(u_0, u_1)$ along the Hugoniot curve $\mathcal{H}_j(u_0)$:

$$D_j^\star(u_0) = \min_{u_1 \in \mathcal{H}_j(u_0)} D(u_0, u_1).$$

Actually the maximum is achieved at the critical value $\mu_j^\star(u_0)$ for the shock speed. Consider also the entropy dissipation as a function of s, say, $d^\star(s)$ defined as

$$d^\star(s) = \max\big\{D_j^\star(u_0) \,|\, u_0 \in \mathcal{U},\ j \in \mathbf{P},\quad \lambda_j\big(u_0, w_j(\mu_j^\star(u_0); u_0)\big) = s\big\}. \tag{2.44}$$

(The value is taken to be $-\infty$ when no u_0 satisfies the constraint.) Note that $D_j^\star$ and $d^\star$ are computable from the expression of the flux f in the examples studied in sections 3–5 below.

Theorem 2.9 below shows that knowing the entropy dissipation of the admissible nonclassical shocks determines a unique solution of the Riemann problem. To solve the Riemann problem, we assume that $\big\{r_k(u, u')\big\}$ is a basis of $\mathbb{R}^N$ for arbitrary u, $u' \in \mathcal{U}$. (This is always true when R is small enough.)

THEOREM 2.9. *Suppose that the system satisfies the condition* (2.10a).

(1) *For $j \in \mathbf{P}$, let $\phi_j : \mathcal{U} \to \mathbb{R}_-$ be a continuous function satisfying*

$$D_j^\star(u_0) \le \phi_j(u_0) \le 0 \qquad \textit{for all } u_0 \in \mathcal{U},\ j \in \mathbf{P}. \tag{2.45}$$

Let $u_0 \in \mathcal{U}$ and $j \in \mathbf{P}$ be given. From the wave set $\mathcal{S}_j(u_0)$, there exists a unique wave curve $\mathcal{W}_j^{nc}(u_0)$ using nonclassical shocks satisfying the kinetic relation (2.42). *For u_l and u_r in $\mathcal{U}$, the Riemann problem* (2.1), (2.11) *admits a unique solution in the class of admissible nonclassical entropy solutions obtained by intersection of the curves $\mathcal{W}_j^{nc}$. Furthermore, the solution depends continuously in the L^1 norm upon its end states.*

(2) *Let $\varphi : \Lambda \to \mathbb{R}_-$ be a Lipschitz continuous function satisfying*

$$d^\star(s) \le \varphi(s) \le 0, \quad \frac{d\varphi}{ds}(s) \le 0 \qquad \textit{for all } s \in \Lambda. \tag{2.46}$$

For the kinetic relation (2.43), *the conclusions are the same as in Case* 1.

(3) *In both Cases* 1 *and* 2 *above, there exist two values $\mu_j^\flat(u_0)$ and $\mu_j^\sharp(u_0)$, with*

$$\mu_j^{\star\star}(u_0) \le \mu_j^{\flat\flat}(u_0) \le \mu_j^\flat(u_0) \le \mu_j^\star(u_0) \le \mu_j^\sharp(u_0) \le \mu_j^{\sharp\sharp}(u_0) \le \mu_j(u_0), \tag{2.47}$$

such that the nonclassical wave curve is composed of the following four pieces:

$$\mathcal{W}_j^{nc}(u_0) = \begin{cases} \mathcal{O}_j(u_0) & \textit{for all } \mu_j \ge \mu_j(u_0), \\ \mathcal{H}_j(u_0) & \textit{for all } \mu_j^\sharp(u_0) \le \mu_j \le \mu_j(u_0), \\ \mathcal{H}_j(u^\flat) & \textit{for all } \mu_j^\flat(u_0) \le \mu_j < \mu_j^\sharp(u_0), \\ \mathcal{O}_j(u^\flat) & \textit{for all } \mu_j \le \mu_j^\flat(u_0), \end{cases}$$

where $u^\flat := w_j\big(\mu_j^\flat(u_0); u_0\big)$. The curve $\mathcal{W}_j^{nc}(u_0)$ is continuous and monotone in the parameter μ_j, and is of class C^2 except at $\mu_j = \mu_j^\sharp(u_0)$, where it is generally only Lipschitz continuous.

We can recover the classical curve $\mathcal{W}_j^c(u_0)$ with the (maximal) choice

$$\phi_j(u_0) = D_j^\star(u_0). \tag{2.48}$$

In that case the classical and nonclassical shocks in the solution have the same propagation speed, and the two waves are indistinguishable in the (x,t) plane. On the other hand it is not possible to use part of the classical wave curve, say, for values $\mu_j > \mu_j^c$, and switch to the nonclassical wave curve, say, for values $\mu_j < \mu_j^c$, at least as far as a Riemann solution depending continuously upon its end states is sought. The latter seems to be a natural requirement, at least in view of the examples studied so far in the literature. Furthermore the classical wave curve $\mathcal{W}_j^c(u_0)$ is always admissible, since Definition 2.8 does not prevent us from solving the Riemann problem by using classical waves only. Therefore, even after imposing the kinetic relation, there exist *two* wave curves to choose from for each nongenuinely nonlinear family, $\mathcal{W}_j^c(u_0)$ and $\mathcal{W}_j^{nc}(u_0)$. It would be interesting to connect this nonuniqueness with instability in solutions to an augmented diffusive-dispersive system with vanishing small-scale parameters.

Proof of Theorem 2.9. Let $u_0 \in \mathcal{U}$ and $j \in \mathbf{P}$ be given. In view of the definition (2.44) of the maximal entropy dissipation and the assumption (2.45), the criterion (2.42) selects a unique nonclassical shock along the Hugoniot curve $\mathcal{H}_j(u_0)$, say, $u^\flat = w_j(\mu_j^\flat(u_0), u_0)$. Once this state is selected, the construction in Theorem 2.6 determines a unique wave curve $\mathcal{W}_j^{nc}(u_0)$ having the form described in item (3) of the theorem. This curve is continuous in the parameter μ_j which by construction is monotone increasing along it. It is of class C^2 at the point $\mu_j(u_0)$ and $\mu_j^\flat(u_0)$ since classical rarefaction curves and shock curves have second-order contact. Finally, along the wave curve, the speeds of the (rarefaction or shock) waves change continuously. To see that, at the point $\mu_j^\sharp(u_0)$, one has to compare, on one hand, the shock speed of the nonclassical shock and, on the other hand, the shock speeds of the nonclassical shock and the classical one. Actually all three terms coincide at $\mu_j^\sharp(u_0)$:

$$\lim_{\substack{\mu_j \to \mu_j^\sharp(u_0) \\ \mu_j > \mu_j^\sharp(u_0)}} \bar\lambda_j(u_0, w_j(\mu_j, u_0)) = \lim_{\substack{\mu_j \to \mu_j^\sharp(u_0) \\ \mu_j < \mu_j^\sharp(u_0)}} \bar\lambda_j\big(u^\flat, w_j(\mu_j; u^\flat)\big) = \bar\lambda_j\big(u_0, u^\flat\big).$$

The continuous dependence of the wave speeds implies the L^1 continuous dependence of the solution. Finally, having constructed the Lipschitz continuous wave curves $\mathcal{W}_j^{nc}$ for $j \in \mathbf{P}$ and the smooth wave curves $\mathcal{W}_j^c$ for $j \notin \mathbf{P}$, and using the condition that $\big\{r_k(u, u')\big\}$ is a basis of $\mathbb{R}^N$ for arbitrary u, u', we can solve the Riemann problem with data in $\mathcal{U}$: combining together the wave curves, we apply the theorem of implicit functions for Lipschitz continuous curves. (For a reference see Isaacson and Temple [29].) The Riemann problem admits a unique solution, at least with data in $B(u_*, R') \subset \mathcal{U}$, with $R' << R$. This proves the items (1) and (3).

In order to use the criterion (2.43), one observes that the entropy dissipation $D(u_0, u_1)$ along the Hugoniot curve—when expressed as a function of the shock speed s—is increasing from its lower value $D_j^\star(u_0)$ at $s^\star = \bar\lambda_j\big(u_0, w_j(\mu_j^\star(u_0); u_0)\big)$ to the value 0 at $s = \bar\lambda_j\big(u_0, w_j(\mu_j^{\flat\flat}(u_0); u_0)\big)$. On the other hand, the function $\varphi(s)$ is assumed to be decreasing in the same interval and by (2.44), (2.46), one has $\varphi(s^\star) \ge d^\star(s) \ge D_j^\star(u_0)$. Thus there exists a unique point $\mu_j = \mu_j^\flat(u_0)$ such that the kinetic relation (2.43) is satisfied. This wave curve shares the same properties as that in the case (2.42). ◻

REMARK 2.10. (1) The assumption that the kinetic function be a decreasing function of the shock speed may be motivated in the following way. Consider a scalar conservation law ($N = 1$) with the flux $f(u) = u^3$ and the entropy $U(u) = u^2/2$. Consider a *linear* relation for nonclassical shocks, say, between the left state u_0 and the right state u_1,

$$u_1 = g(u_0) := \beta\, u_0. \tag{2.49}$$

According to the theory in this section, one must have $\beta \in (-1, -1/2)$. Plugging (2.49) into the definition of the entropy dissipation $D(u_0, u_1)$ the kinetic relation corresponding to (2.49) can be computed:

$$\begin{aligned}\varphi(s) := D\big(u_0, g(u_0)\big) &= -\,(1+\beta)(1-\beta)^3\, u^4 \\ &= -\,(1+\beta)(1-\beta)^3(1+\beta+\beta^2)^{-2}\, s^2,\end{aligned}$$

which indeed is a decreasing function of s in the interesting range $s > 0$.

(2) In the classical solution, the value of the intermediate state (if any) in the Riemann solution varies continuously as $u_1 \in \mathcal{W}_j^c(u_0)$ describes the wave curve; the solution in the (x,t) plane varies continuously in the L^1 norm and its total variation is a continuous function of the end points. For the nonclassical wave curve, the wave speeds only are continuous, and the total variation of the Riemann solution is not a continuous function of the endpoints. □

To conclude this section, we explain how to determine the kinetic function, needed in (2.42) or (2.43). Consider a sequence of solutions u^ϵ to a regularized version of (2.1) of the form (2.2). Assume for the sake of this presentation that the u^ϵ remain bounded in the total variation norm and converge to a limiting solution u to (2.1), (2.5). Suppose also that the system admits an entropy pair that is compatible with the regularization (2.2). We know that the entropy inequality (2.5) is too lax to guarantee uniqueness for the Riemann problem. Another Rankine–Hugoniot relation, in addition to the set of conservation laws contained in (2.1), is in principle sufficient to select a unique nonclassical solution.

The concepts of entropy and entropy dissipation are fundamental in the theory of hyperbolic conservation laws. It seems mathematically natural to go beyond the entropy *inequality* (2.8) and instead write the entropy *balance*:

$$\partial_t U(u) + \partial_x F(u) = \mu_U \le 0. \tag{2.50}$$

Here μ_U is a bounded, nonpositive Borel measure, which provides partial information on the small-scale effects in the regularization sequence that generated the solution u. The dissipation measure generated by a regularization (2.2) satisfying the condition (2.3) is

$$\mu_U := w - \star \lim_{\epsilon \to 0}\ \epsilon\, \partial_x u_\epsilon^T \nabla^2 U(u_\epsilon) B_1(u_\epsilon) \partial_x u_\epsilon. \tag{2.51}$$

Since u solves (2.1), the measure μ_U has its support included in the union of the set of points of approximate discontinuity of u.

The mass of the measure along the curve of discontinuity is the entropy dissipation $D(.,.)$.

Of course the knowledge of the measure μ_U in (2.50) is required only for nonclassical shocks, since the propagation of a *classical* shock is uniquely determined by the Rankine–Hugoniot relation

$$-\bar\lambda_j(u_0, u_1)\,\big(u_1 - u_0\big) + f(u_1) - f(u_0) = 0$$

and the entropy *inequality*

$$-\bar\lambda_j(u_0,u_1)\ \big(U(u_1)-U(u_0)\big)+F(u_1)-F(u_0)\le 0.$$

The entropy dissipation measure μ_U for a *nonclassical* shock, as determined by (2.51), in general, will depend upon the left state u_0 and the shock speed, $s=\bar\lambda_j(u_0,u_1)$. The kinetic relation generated by (2.2) can be determined, at least at a formal level, from an analysis of admissible traveling wave solutions to (2.2). Different approximations to (2.1) will result, in general, in different kinetic relations. Consider a traveling wave solution $u_\epsilon(x,t)=w((x-s\,t)/\epsilon)$ to (2.2), that is a solution to the ordinary differential equation in $\xi=(x-s\,t)/\epsilon$

$$-s\,w'+f(w)'=\big(B_1(w)\,w'\big)'+\big(B_2(w)\,w''\big)' \tag{2.52}$$

satisfying the following boundary conditions

$$\begin{aligned}&\lim_{\xi\to-\infty} w(\xi)=u_0, \qquad \lim_{\xi\to\infty} w(\xi)=u_1,\\ &\lim_{\xi\to\pm\infty} w'(\xi)=0, \quad \lim_{\xi\to\pm\infty} w''(\xi)=0.\end{aligned} \tag{2.53}$$

The equation (2.52) can be integrated once:

$$-s\,(w-u_0)+f(w)-f(u_0)=B_1(w)\,w'+B_2(w)\,w''. \tag{2.54}$$

The internal structure of the nonclassical shock is represented by the trajectory $\xi\to w(\xi)$, which can be used to determine the entropy dissipation measure. Namely, at the hyperbolic level we have

$$\begin{aligned}D(u_0,u_1)&=-\bar\lambda_j(u_0,u_1)\,\big(U(u_1)-U(u_0)\big)-F(u_1)+F(u_0)\\ &=\int_{\mathbb{R}}\nabla U(w)\cdot\big(-\bar\lambda_j(u_0,u_1)+Df(w)\big)\,w'\,d\xi\\ &=-\int_{\mathbb{R}} w'\cdot\nabla^2U(w)\cdot\big(-\bar\lambda_j(u_0,u_1)\,(w-u_0)+f(w)-f(u_0)\big)\,d\xi.\end{aligned}$$

Using (2.54) for the traveling wave and the conditions (2.3), we obtain

$$D(u_0,u_1)=-\int_{\mathbb{R}}(w')^T\nabla^2U(w)B_1(w)\,w'\,d\xi\le 0. \tag{2.55}$$

In the examples arising in continuum mechanics, at least, the entropy dissipation for a nonclassical shock, computed from (2.55), can be expressed as a function of the state u_0 (or, equivalently, u_1). (See also section 4.1.)

3. Nonclassical shocks in elastodynamics (1). We now turn to a model arising in the theory of elastic materials, which is strictly hyperbolic and admits two nongenuinely nonlinear characteristic fields. This section restricts attention to the Riemann problem and extends the analysis of section 2 to arbitrarily large initial data.

3.1. Preliminaries. Consider the system of elastodynamics

$$\begin{aligned}\partial_t v-\partial_x\sigma(w)&=0,\\ \partial_t w-\partial_x v&=0,\end{aligned} \tag{3.1}$$

where the real-valued functions v and w represent the velocity and gradient deformation, respectively. The stress-strain law is assumed to have the form

$$\sigma(w) = w^3 + m^2\, w, \qquad m > 0. \tag{3.2}$$

The focus here is on Riemann data

$$v(x,0),\ w(x,0) = \begin{cases} v_l, w_l, & x < 0, \\ v_r, w_r, & x > 0, \end{cases} \tag{3.3}$$

for constants $v_l, w_l, \dots$. We note that (3.1)–(3.2) is invariant under any of the transformations:

$$w \to -w, \quad v \to -v, \tag{3.4i}$$

$$v \to v + \bar{v} \quad \text{(for any constant } \bar{v}\text{)}, \tag{3.4ii}$$

$$x \to -x, \quad v \to -v. \tag{3.4iii}$$

We may write (3.1) in the general form (2.1) by setting $u = (v, w)$, $f(u) = -\big(\sigma(w), v\big)$. The system is strictly hyperbolic with eigenvalues $\lambda_1(v,w) = -c(w) < 0 < \lambda_2(v,w) = c(w)$, where the sound speed is defined by $c(w) = \sqrt{3\, w^2 + m^2}$. Since the wave speeds are independent of v, the notation $\lambda_1(w) = -c(w)$ and $\lambda_2(w) = c(w)$ is also used. The wave speeds are strictly separated: they keep different signs and are bounded away from zero. The right eigenvectors may be chosen as $r_i(v,w) = (\pm c(w), 1)$ for $i = 1, 2$.

We consider the wave curves for the system (3.1). The Hugoniot locus $\mathcal{H}_1(v_0, w_0)$ consists of all the states (v_1, w_1) connected to (v_0, w_0) on the left by a discontinuity with speed $s < 0$. Similarly, $\mathcal{H}_2(v_0, w_0)$ corresponds to the discontinuities with speed $s > 0$. The Rankine–Hugoniot condition gives

$$-s = \frac{v - v_0}{w - w_0} = \frac{\sigma(w) - \sigma(w_0)}{v - v_0}. \tag{3.5}$$

A discontinuity connecting (v_0, w_0) to (v, w) therefore travels with speed $s = \pm\, \bar{c}(w_0; w)$, where we use the notation $\bar{c}(w_0; w) = \sqrt{w_0^2 + w_0\, w + w^2 + m^2}$. Observe that $\bar{c}(w; w) = c(w)$. We emphasize that $\bar{c}(w_0; w)$ is the magnitude of the shock speed and is always positive. From (3.5) we obtain

$$\mathcal{H}_1(v_0, w_0) \;=\; \Big\{ v \in \mathbb{R} \,|\, v - v_0 \;=\; \bar{c}(w_0; w)\,(w - w_0) \Big\}, \tag{3.6}$$

$$\mathcal{H}_2(v_0, w_0) \;=\; \Big\{ v \in \mathbb{R} \,|\, v - v_0 \;=\; -\,\bar{c}(w_0; w)\,(w - w_0) \Big\}. \tag{3.7}$$

In addition, the rarefaction waves are based on the integral curves of the vector fields r_j:

$$\mathcal{O}_1(v_0, w_0) \;=\; \Big\{ v \in \mathbb{R} \,|\, v - v_0 \;=\; \int_{w_0}^{w} c(z)\, dz \Big\}, \tag{3.8}$$

$$\mathcal{O}_2(v_0, w_0) \;=\; \Big\{ v \in \mathbb{R} \,|\, v - v_0 \;=\; - \int_{w_0}^{w} c(z)\, dz \Big\}. \tag{3.9}$$

The system (3.1)–(3.2) is not genuinely nonlinear since $\nabla\lambda_1(w)\cdot r_1(w) = -\,3\,w/c(w)$ and $\nabla\lambda_2(w)\cdot r_2(w) \;=\; 3\,w/c(w)$, which vanish on the (one-dimensional) manifold $\mathcal{M} \;= \mathcal{M}_1 = \mathcal{M}_2 = \big\{(v, w)\,|\, w = 0\big\}$. In order to uniquely solve the Riemann problem, we now apply appropriate entropy criteria. Away from the line $w = 0$, the system has two genuinely nonlinear fields; therefore, for shocks with small amplitude, the Lax shock inequalities may be used.

3.2. Liu's construction of a unique solution. Here we briefly summarize the Liu's construction for the system (3.1). For a point (v, w) in $\mathcal{H}_1(v_0, w_0)$, the Liu entropy criterion implies the Lax shock inequalities, $-c(w_0) \geq -\bar{c}(w_0; w) \geq -c(w)$, and, as pointed out in section 2, is actually equivalent to them since the stress-strain relation has a single inflexion point. Defining

$$\kappa = w/w_0, \tag{3.10}$$

and using the expressions for $c(w)$ and $\bar{c}(w_0; w)$, one sees that the admissible region for $\mathcal{H}_1(v_0, w_0)$ consists of all (v, w) with

$$\kappa \in (-\infty, -2] \cup [1, +\infty)\,. \tag{3.11}$$

For $\mathcal{H}_2(v_0, w_0)$, the shock speed is positive and the Liu criterion leads to the interval

$$\kappa \in [-1/2, 1]\,. \tag{3.12}$$

Note in passing that the intervals found in (3.11) and (3.12) are independent of m. We now utilize (3.11)–(3.12) and construct the classical wave curves $\mathcal{W}_j^c(v_0, w_0)$. Consider a point (v_0, w_0) with $w_0 > 0$. By (3.4ii), $\mathcal{W}_j^c(v_0', w_0)$ for $v_0' \neq v_0$ is a suitable translate of $\mathcal{W}_j^c(v_0, w_0)$, while (3.4i) allows the construction for $w_0 > 0$ to be simply extended to the case $w_0 < 0$.

The wave curves are easily defined locally. These curves are $\mathcal{H}_1(v_0, w_0)$, $\mathcal{O}_1(v_0, w_0)$, $\mathcal{H}_2(v_0, w_0)$, and $\mathcal{O}_2(v_0, w_0)$ for values $w > w_0$, $w < w_0$, $w < w_0$, and $w > w_0$, respectively. Note that since $\nabla\lambda_i \cdot r_i \;=\; \pm 3\,w/c(w)$ changes signs only along curves crossing $w = 0$, we see immediately that the curves $\mathcal{H}_1(v_0, w_0)$ and $\mathcal{O}_2(v_0, w_0)$ may be extended to all points (v, w) such that $w > w_0$. These two curves correspond to functions $w \to v(w)$ that are increasing and decreasing, respectively, according to the formulas (3.6) and (3.9).

We now turn to those wave curves which cross the line $w = 0$. For $0 < w \leq w_0$, we have $\nabla\lambda_i \cdot r_i \;<\; 0$, so that all points (v, w) in this region, lying on $\mathcal{O}_1(v_0, w_0)$, may be arrived at by a single 1-rarefaction. This construction changes for $w <$ 0: when $-2\,w_0 \;<\; w \;<\; 0$, there is a critical point on the rarefaction curve, say, $(v_*, w_*) \in \mathcal{O}_1(v_0, w_0)$ with $w_* > 0$, for which $\bar{c}(w_0; , w_*) = c(w_*)$. This point satisfies $w_* = -w/2$.

According to the Liu criterion, in order to reach a point (v, w) from (v_0, w_0), having $-2\,w_0 < w < 0$, the solution proceeds along $\mathcal{O}_1(v_0, w_0)$ until it reaches (v_*, w_*), at which point it jumps on $\mathcal{H}_1(v_*, w_*)$ to (v, w). We denote this composite curve by

$$\begin{aligned} \mathcal{K}_1(v_0, w_0) = \Big\{ (v, w)\,|\; & \text{ there exists } \ (v_*, w_*) \in \mathcal{O}_1(v_0, w_0), \quad 0 < w_* < w_0, \\ & \text{such that } \ w = -2\,w_* \ \text{ and } (v, w) \in \mathcal{H}_1(v_*, w_*) \Big\}. \end{aligned} \tag{3.13}$$

It may be shown that along $\mathcal{K}_1(v_0, w_0)$, v is monotone increasing with w. When $w \le -2\,w_0$, the curve $\mathcal{K}_1(v_0, w_0)$ may be continued, by virtue of (3.11), as a single 1-shock, i.e., $(v, w) \in \mathcal{H}_1(v_0, w_0)$, when $w \le -2\,w_0$ and v is thus given by the Rankine–Hugoniot relation (3.5). Note that $\mathcal{K}_1(v_0, w_0)$ joins $\mathcal{H}_1(v_0, w_0)$ at the point $(v_{**}, w_*) = (v_{**}, -2\,w_0) \in \mathcal{H}_1(v_0, w_0)$, and $\mathcal{K}_1(v_0, w_0)$ joins $\mathcal{O}_1(v_0, w_0)$ at the point $(0, v_0) \in \mathcal{O}_1(v_0, w_0)$. We summarize in the next lemma.

LEMMA 3.1. *The classical* 1*-wave curve from a point* $(v_0, w_0), w_0 > 0$*, is the union of four pieces:*

$$\mathcal{W}_1^c(v_0, w_0) = \begin{cases} \mathcal{H}_1(v_0, w_0) & \text{for } w > w_0, \\ \mathcal{O}_1(v_0, w_0) & \text{for } 0 \le w \le w_0, \\ \mathcal{K}_1(v_0, w_0) & \text{for } -2\,w_0 \le w < 0, \\ \mathcal{H}_1(v_0, w_0) & \text{for } w < -2\,w_0. \end{cases}$$

It is a monotone increasing curve of class $\mathcal{C}^\infty$*, extending from* $(v, w) = (-\infty, -\infty)$ *to* $(v, w) = (+\infty, +\infty)$*.*

The construction of the 2-wave curve is similar and we summarize its properties as follows.

LEMMA 3.2. *The classical* 2*-wave curve from* (v_0, w_0)*, with* $w_0 > 0$*, is the union of three pieces:*

$$\mathcal{W}_2^c(v_0, w_0) = \begin{cases} \mathcal{O}_2(v_0, w_0) & \text{for } w > w_0, \\ \mathcal{H}_2(v_0, w_0) & \text{for } -w_0/2 \le w \le w_0, \\ \mathcal{O}_2(v_*, w_*) & \text{for } w < -w_0/2, \end{cases}$$

where $(v_*, w_*) \in \mathcal{H}_2(v_0, w_0)$ *and* $w_* = -w_0/2$*. It is a monotone decreasing curve of class* $\mathcal{C}^\infty$*, extending from* $(v, w) = (+\infty, -\infty)$ *to* $(v, w) = (-\infty, +\infty)$*.*

The infinite extent in v of the 2-wave curve follows from the fact that the integral curves in (3.9) have no horizontal asymptotes. This completes the construction of the wave curves based on the Liu criterion. A unique solution exists for arbitrary Riemann data. It can be checked that this solution depends continuously upon its initial states.

3.3. Two-parameter family of nonclassical entropy solutions. We apply Definition 2.1 to the system (3.1) and construct a two-parameter family of solutions. Definition 2.1 is based on a specific convex entropy pair, which we take here to be

$$U(v, w) = \frac{v^2}{2} + \frac{w^4}{4} + m^2\,\frac{w^2}{2}, \qquad F(v, w) = -v\,\sigma(w). \tag{3.14}$$

This choice is based on the physically motivated regularization studied in section 4. A brief computation leads to the following formula for the entropy dissipation:

$$D(v_-, w_-; v_+, w_+) = -s\big(\bar{w}(m^2 + \bar{w^2})\,[w] + \bar{v}\,[v]\big)\big(m^2\bar{w} + \bar{w^3}\big)\,[v] - \bar{v}\,[\sigma(w)] \tag{3.15}$$

with $[\alpha] = \alpha_+ - \alpha_-$ and $\bar{\alpha} = (\alpha_+ + \alpha_-)/2$. We now substitute the Rankine–Hugoniot relations (3.5) to get

$$D(v_-, w_-; v_+, w_+) = \bar{w}\,\bar{w^2}\ [v] - \bar{w^3}\ [v] = -\frac{1}{2}\bar{w}\ [w]^2\ [v]\,.$$

The entropy inequality (2.8), (3.14) therefore reduces to

$$\bar{w}\ [v] \ge 0 \tag{3.16}$$

(for $[w] \neq 0$). If we now utilize (3.6)–(3.7) for $\mathcal{H}_1(v_-, w_-)$ and $\mathcal{H}_2(v_-, w_-)$, we find that

$$D(v_-, w_-; v_+, w_+) = \frac{s}{2}\,[w]^3\,\bar{w}. \tag{3.17}$$

We recall that $s < 0$ for $\mathcal{H}_1(v_-, w_-)$ and $s > 0$ for $\mathcal{H}_2(v_-, w_-)$. From now on we express the entropy dissipation as a function of w_- and w_+ alone: $D(w_-; w_+)$. The admissible nonclassical shocks from (v_-, w_-) to (v_+, w_+) must therefore satisfy

$$|w_+| \geq |w_-| \quad \text{along} \quad \mathcal{H}_1(v_-, w_-), \qquad |w_+| \leq |w_-| \quad \text{along} \quad \mathcal{H}_2(v_-, w_-). \tag{3.18}$$

Since $\mathcal{H}_1(v_-, w_-)$, restricted by the condition (3.8), forms a nonconnected set, we denote the portion of $\mathcal{H}_1(v_-, w_-)$ with $w_+ \geq w_-$ by $\mathcal{H}_1^+(v_-, w_-)$, while that portion having $w_+ \leq -w_-$ will be denoted by $\mathcal{H}_1^-(v_-, w_-)$.

We now introduce solutions containing nonclassical shocks. Consider a point (v_0, w_0) with $w_0 > 0$. Owing to transformations (3.4), a translation in v_0 simply effects the same translation in the entire solution; furthermore, we can obtain the wave curves for $w_0 < 0$ by switching the signs of both w and v. We begin by discussing the 1-wave curves. As in the classical case, the solution may leave (v_0, w_0) along $\mathcal{O}_1(v_0, w_0)$ and proceed until it reaches the point $(\tilde{v}, \tilde{w})$ with $\tilde{w} = 0$.

LEMMA 3.3. *From any point $(v_1, w_1) \in \mathcal{O}_1(v_0, w_0)$, with $0 < w_1 < w_0$, it is possible to jump to a point $(v_2, w_2) \in \mathcal{H}_1^-(v_1, w_1)$ with $w_2 \in [-2\,w_1, -w_1]$.*

Proof of Lemma 3.3. By (3.18), one has $w_2^2 - w_1^2 \geq 0$. In addition, for the shock to follow the rarefaction, one needs $0 > -\bar{c}(w_1; w_2) \geq \lambda_1(w_1)$, so that $(w_2 + 2\,w_1)(w_2 - w_1) \leq 0$. The intersection of these two regions is the interval $-2\,w_1 \leq w_2 \leq -w_1$. Of the points w_2 in this interval, only the right-hand boundary $w_2 = -2\,w_1$ corresponds to a classical shock. ∎

As (v_1, w_1) varies from (v_0, w_0) to $(\tilde{v}, 0)$, along $\mathcal{O}_1(v_0, w_0)$, the set of image points, $\{(v_2, w_2)\}$, of these nonclassical shocks covers a bounded region. We refer to these wave fans as $\mathcal{O}_1$-$\mathcal{H}_1^-$ nonclassical solutions. From (3.18), it is also possible to leave (v_0, w_0) by a shock, i.e., to jump to $(v_1, w_1) \in \mathcal{H}_1(v_0, w_0)$ for $|w_1| \geq |w_0|$. We note that for $w_1 \geq w_0$ and for $w_1 \leq -2\,w_0$, these are classical shocks. In addition we have the following.

LEMMA 3.4. *From a point $(v_1, w_1) \in \mathcal{H}_1^+(v_0, w_0)$, it is possible to jump via a nonclassical shock to $(v_2, w_2) \in \mathcal{H}_1^-(v_1, w_1)$ with $-w_0 - w_1 \leq w_2 \leq -w_1$. The points with $w_2 = -w_0 - w_1$ lie on $\mathcal{H}_1(v_0, w_0)$. The region containing a classical shock along $\mathcal{H}_1^+(v_0, w_0)$, followed by a nonclassical shock along $\mathcal{H}_1^-(v_1, w_1)$, extends indefinitely to the left in w_2, and down in v_2.*

Proof of Lemma 3.4. Once again (3.18) gives $|w_2| \geq |w_1|$, and for the nonclassical shock to follow the classical one, one must also have $-\bar{c}(w_1; w_2) \geq -\bar{c}(w_0; w_1)$. Manipulating the expression for s leads to $|2\,w_2 + w_1| \leq |2\,w_0 + w_1|$, and, using the fact that $0 < w_0 < w_1$, this has the solution $-w_0 - w_1 \leq w_2 \leq w_0$. Combining this with the entropy inequality leads to $-w_0 - w_1 \leq w_2 \leq -w_1$.

For $w_2 = -w_1 - w_0$, one has $-\bar{c}(w_0; w_1) = -\bar{c}(w_0; w_2)$ and, by using the Rankine–Hugoniot condition (3.5), one can show that $(v_2, w_2) \in \mathcal{H}_1^-(v_0, w_0)$. Since $w_2 \leq -2w_0$, the point (v_2, w_2) is in the classical portion of this Hugoniot curve.

According to (3.18), a point $(v_1, w_1) \in \mathcal{H}_1^+(v_0, w_0)$ may have w_1 arbitrarily large and positive, so that $w_2 \leq -w_1$ can be arbitrarily large and negative. A calculation using the Hugoniot curve shows that

$$\begin{aligned} v_2 &= v_0 + \bar{c}(w_0; w_1)\,(w_1 - w_0) + \bar{c}(w_1; w_2)\,(w_2 - w_1) \\ &\leq v_0 + \bar{c}(w_0; w_1)\,(w_1 - w_0) - 2\,w_1\,c(w_1), \end{aligned}$$

so that as $w_1 \to +\infty$ with (v_0, w_0) fixed, we have $v_2 \leq -w_1^2\,(1 + o(1)) \to -\infty$, so the upper boundary, and hence the entire region, tends to negative infinity, as $w_2 \to -\infty$. There is no horizontal asymptote. □

We refer to these 2-wave fans as $\mathcal{H}_1^+$-$\mathcal{H}_1^-$ nonclassical solutions. A similar argument shows that no $\mathcal{O}_1$-$\mathcal{H}_1^-$ or $\mathcal{H}_1^+$-$\mathcal{H}_1^-$ wave fan may be connected to *additional* states by a 1-wave.

We now turn to the 2-wave family, again taking (v_0, w_0) with $w_0 > 0$. In this case, $\lambda_2(w) = \sqrt{3\,w^2 + m^2}$ is increasing with w, so that any point $(v_1, w_1) \in \mathcal{O}_2^+(v_0, w_0)$, i.e., with $w_1 \geq w_0$, may be connected to (v_0, w_0) via a 2-rarefaction. We may not continue from (v_1, w_1) to a point $(v_2, w_2) \in \mathcal{H}_2(v_1, w_1)$, since the entropy inequality, which gives $|w_2| \leq |w_1|$, and the proper ordering of wave speeds, which implies $w_2 \geq w_1$, have only the degenerate point $(v_2, w_2) = (v_1, w_1)$ of intersection. If instead we leave (v_0, w_0) via $\mathcal{H}_2(v_0, w_0)$, the entropy inequality permits us to proceed to the left, until we reach $(\tilde{v}, \tilde{w}) \in \mathcal{H}_2(v_0, w_0)$ with $\tilde{w} = -w_0$. Note that this shock is nonclassical for $-w_0 \leq w_1 < -w_0/2$.

LEMMA 3.5. *For $(v_1, w_1) \in \mathcal{H}_2(v_0, w_0)$ with $-w_0 \leq w_1 \leq -w_0/2$, it is possible to connect to a point $(v_2, w_2) \in \mathcal{H}_2(v_1, w_1)$ with $w_1 \leq w_2 \leq -w_0 - w_1$. This part of the curve $\mathcal{H}_2(v_1, w_1)$ extends until it reaches a point $(v_2, w_2) = (v_2, -w_0 - w_1) \in \mathcal{H}_2(v_0, w_0)$.*

Proof of Lemma 3.5. Starting from a point $(v_1, w_1) \in \mathcal{H}_2(v_0, w_0)$, we proceed with a 2-shock on the right, to a point $(v_2, w_2) \in \mathcal{H}_2(v_1, w_1)$. The entropy inequality forces $|w_2| \leq |w_1|$. In addition, the requirement that $c(w_1; w_2) \geq c(w_0; w_1) \geq 0$ implies that $|2\,w_2 + w_1| \geq |2\,w_0 + w_1|$. Since $w_0 \geq |w_2|$, we must take $w_1 \leq 0$. The condition then becomes $|2\,w_2 - |w_1|| \geq 2\,w_0 - |w_1|$, so that w_2 must also be nonpositive. Some manipulation gives $w_2 \leq -w_0 - w_1 \leq 0$, so that in combination with (3.18) we have $w_2 \in [w_1, -w_0 - w_1]$, and w_1 has the restriction that $w_1 \leq -w_0 - w_1$, so that $w_1 \leq -w_0/2$. This leads to $w_1 \in [-w_0, -w_0/2]$. At the right-hand end of this interval, $w_1 = -w_0/2$, the shock is classical. □

As w_1 varies about the interval $[-w_0, -w_0/2]$, the set $\{(v_2, w_2)\}$ of image points attainable from (v_0, w_0) by a nonclassical shock, followed by a second shock, fill up a bounded region. This second shock is always a classical one, across which w does not change signs. Points on this second shock may not, therefore, be connected to a further rarefaction or shock wave. We now consider rarefaction waves originating at a point on $\mathcal{H}_2(v_0, w_0)$.

LEMMA 3.6. *A point $(v_1, w_1) \in \mathcal{H}_2(v_0, w_0)$, with $-w_0 \leq w_1 \leq -w_0/2$, may be connected to any point $(v, w) \in \mathcal{O}_2(v_1, w_1)$ having $w \leq w_1$.*

Proof of Lemma 3.6. Since $\lambda_2(w)$ is increasing for $|w|$ increasing, if points (v_1, w_1) can be found so that $c(w_1) \geq c(w_0; w_1) \geq 0$, then the rarefaction curves $\mathcal{O}_2(v_1, w_1)$ may be continued indefinitely to the left. The condition on wave speeds reduces to $(2w_1 + w_0)(w_1 - w_0) \geq 0$, so that we must have $w_1 \leq -w_0/2$. Thus any (v_1, w_1) with $w_1 \in [-w_0, -w_0/2]$ can serve as the origin of a 2-rarefaction. Note that the classical shock-rarefaction occurs for $w_1 = -w_0/2$. □

From (3.8)–(3.9), all of the (classical and nonclassical) integral curves have $v \to +\infty$ as $w \to -\infty$. As w_1 varies from $-w_0/2$ to $-w_0$, the set of points that may be reached by a nonclassical 2-shock, followed by a rarefaction, forms an unbounded

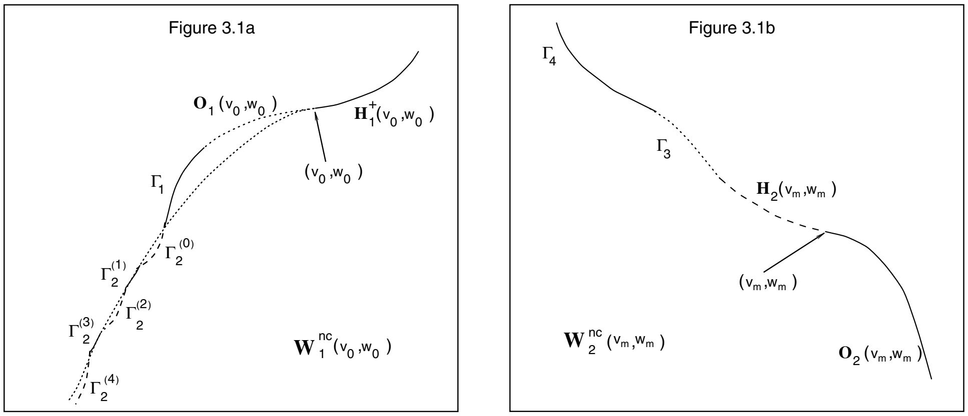


FIG. 3.1. *Wave curve for* (a) *the* 1*-wave family,* (b) *the* 2*-wave family.*

strip in the (v, w)-plane. These rarefaction curves may not be further joined to 2-shock curves, since the entropy inequality and the proper wave speed ordering (the shock must travel faster than the maximum wave speed of the rarefaction fan) lead to incompatible intervals in w. We summarize the above results in this subsection by stating the following.

THEOREM 3.7. *The solutions to* (3.1)–(3.3) *satisfying a single entropy inequality form a one-parameter family in each of the two characteristic fields. The shock speeds* s_1 *and* s_2 *of the nonclassical shocks in the* 1*- and* 2*-wave families, respectively, may be used as the parameters. Given a left-hand state* (v_0, w_0) *and denoting the left-hand state of the nonclassical shock by* (v_-, w_-)*, there are nonclassical solutions in the* 1*-family for* s_1 *satisfying*

$$max\left\{ -\sqrt{w_0^2 + w_0\, w_- + w_-^2 + m^2}\,,\, -\sqrt{3\, w_-^2 + m^2} \right\} \;\le s_1 \;\le\; -\sqrt{w_-^2 + m^2},$$

and in the 2*-family for* s_2 *satisfying*

$$\sqrt{(3/4) w_-^2 + m^2} \;\le\; s_2 \;\le\; \sqrt{w_-^2 + m^2}.$$

3.4. Unique admissible nonclassical entropy solution. In this section, we construct the nonclassical wave curves $\mathcal{W}_j^{nc}(v_0, w_0)$, $j = 1, 2$, displayed in Figure 3.1. For a solution connecting $u_0 = (v_0, w_0)$ to $u_1 = (v_1, w_1)$, we label the successive states, according to increasing wave speed, by $u_0, u_{i_1} = (v_{i_1}, w_{i_1}), u_m = (v_m, w_m), u_{i_2} = (v_{i_2}, w_{i_2})$, and u_1. For classical shocks or rarefactions in the 1-wave and 2-waves curves, the points u_{i_1} and u_{i_2}, respectively, degenerate into u_0 and u_1, respectively. Nonclassical 1-shocks always connect u_{i_1} to a range of u_m, while nonclassical 2-shocks join u_m to a range of u_{i_2}. In the classical cases where shocks are attached to rarefactions, one always has $w_m = -2 w_{i_1}$ and/or $w_{i_2} = -w_m/2$.

Depending on u_0 and u_1, a nonclassical shock may appear in either $\mathcal{H}_1$, or $\mathcal{H}_2$, or both. To select the unique nonclassical shock from among the one-parameter families of solutions found in the previous subsection, we will utilize a *kinetic relation*, stipulating that any nonclassical 2-shocks from u_m to u_{i_2} must satisfy

$$D(w_m; w_{i_2}) = \varphi(s), \tag{3.19}$$

where φ is the kinetic function depending upon the shock speed s. For a given left-hand state (v_m, w_m), we show that the kinetic relation produces a unique right-hand state (v_{i_2}, w_{i_2}), where w_{i_2} depends only on w_m, and not on v_m, say, $w_{i_2} = g(w_m)$. From Lemmas 3.3 and 3.4, $w_{i_2} \in [-w_m, -w_m/2)$.

In order to select a unique nonclassical shock in the 1-wave family, a symmetry of system (3.1) is utilized: the nonclassical 1-shock from u_{i_1} to u_m is selected from among the possible nonclassical 1-shocks from u_{i_1}, if the kinetic relation (3.19) is satisfied for $w_{i_2} = w_{i_1}$. We begin with the kinetic relation for $\mathcal{H}_2$-shocks, divorced from their role in the solution of the Riemann problem, and consider nonclassical $\mathcal{H}_2$-shocks from (v_-, w_-) to (v_+, w_+).

THEOREM 3.8. *Denote by* $I = [m, \infty)$ *the range of positive shock speeds, s, and consider a kinetic function* $\varphi(s)$ *having the following properties:*

$$\frac{d\varphi}{ds} < 0 \quad \textit{for} \quad s \in I, \tag{3.20}$$

$$\varphi(s) \leq 0 \quad \textit{for all} \quad s \in I, \tag{3.21}$$

$$\varphi(s) \geq -\frac{3}{4} s\,(s^2 - m^2)^2 \quad \textit{for } s \in I. \tag{3.22}$$

Then the kinetic relation (3.19) *selects a unique value of the right-hand state* $w_+ = g(w_-)$*, from among the nonclassical shocks in* $\mathcal{H}_2(v_-, w_-)$.

Proof of Theorem 3.8. Without loss of generality, we take $w_- > 0$. From (3.17), we have

$$D(w_-; w_+) = \bar{c}(w_-; w_+)(w_+ - w_-)^3(w_+ + w_-)/4, \tag{3.23}$$

and, from the previous subsection, $-w_- \leq w_+ < -w_-/2$ for the $\mathcal{H}_2$ nonclassical shock. The following calculation shows that the entropy dissipation of (3.23) is monotone in w_+:

$$\frac{\partial D(w_-; w_+)}{\partial w_+} = (2\,w_+ + w_-)\,(5\,w_+^2 + 4\,w_+ w_- + 3\,w_-^2 + 4\,m^2)\,\frac{(w_+ + w_-)^2}{2\,\bar{c}(w_-; w_+)}. \tag{3.24}$$

We rewrite the second factor in (3.24) as $3\,w_+^2 + w_-^2 + 2\,(w_+ + w_-)^2 + 4\,m^2 > 0$. So only the first factor may change sign, and therefore

$$\partial D/\partial w_+ < 0 \quad \text{along} \quad \mathcal{H}_2(v_-, w_-) \quad \text{for} \quad w_+ < -w_-/2, \tag{3.25}$$

so that D is monotone decreasing in w_+ for nonclassical 2-shocks.

For fixed $w_- > 0$ and $w_+ < -w_-/2$,

$$\frac{\partial c(w_-; w_+)}{\partial w_+} = \frac{2w_+ + w_-}{2\,\bar{c}(w_-; w_+)} < 0, \tag{3.26}$$

so that combined with (3.25), this shows that in the region of admissible nonclassical 2-shocks, D is increasing with s. Therefore by (3.20), the functions $D(w_-; w_+)$ and $\varphi(s)$ can have at *most* one intersection point.

We now verify that conditions (3.21) and (3.22) ensure one such intersection. Initially, by (3.18), we have $D(w_-; w_+) \leq 0$. Condition (3.21) is therefore a natural upper bound on φ. In addition, the maximum negative entropy dissipation for a given w_- occurs at the "classical" endpoint, $w_+ = -w_-/2$, of the admissible nonclassical interval. At this point, $s = \sqrt{3\,w_-^2/4 + m^2}$ and by (3.24),

$$D(w_-; -w_-/2) = s\,(-3\,w_-/2)^3\,(w_-/2)/4 \\ = -\frac{3}{4} s\,(s^2 - m^2)^2 . \tag{3.27}$$

Thus, if $\varphi(s)$ remains within the bounds (3.21) and (3.22), the kinetic relation (3.19) must have a unique solution. This completes the proof of Theorem 3.8. ∎

Remark. The construction of Theorem 3.8 cannot be extended to cover the nonclassical 1-shocks, i.e., to the interval $s \in (-\infty, -m]$, as the following argument demonstrates. For the admissible, nonclassical 1-shock region, $-2w_- < w_+ \le -w_-$, one finds again that the entropy dissipation D is monotonically increasing with s. This compels us to take $\varphi'(s) < 0$ in $(-\infty, -m]$. On the other hand, we also require $\varphi(s) \le 0$, and $\varphi(s) \ge 3s(s^2 - m^2)/4$, with this latter function increasing to zero at the right-hand endpoint of the interval. No kinetic function ϕ can possibly satisfy this combination of constraints over the whole interval of s.

We therefore abandon the idea of having independently prescribed kinetics for each of the families of waves. Instead, we will show existence and uniqueness for the Riemann problem under an assumption of *symmetric kinetics*. With symmetric kinetics, *a nonclassical $\mathcal{H}_1$-shock from w_{i_1} to w_m is selected if the kinetic relation for $\mathcal{H}_2$ selects a shock from w_m to $w_{i_2} = w_{i_1}$.* For the case of nonclassical shocks in both families, this assumption results in the two nonclassical shocks being mirror images of each other across the w-axis in the (x, w)-plane. We will see in section 4 that a numerical scheme for (3.1) produces such symmetric nonclassical shocks.

We motivate a symmetric choice of w_{i_1} by noting that system (3.1) is invariant under the transformation $x \to -x, \quad v \to -v$. Thus to any nonclassical 2-shock from (v_m, w_m) to (v_{i_2}, w_{i_2}), there corresponds a nonclassical 1-shock from (v_{i_1}, w_{i_1}) to (v_m, w_m) with $w_{i_1} = w_{i_2}$. These shocks are actually antisymmetric in v and have $v_{i_1} = 3v_{i_2} - 2v_m$. Whether or not such nonclassical shocks are admissible depends on the relative values of w_0 and w_{i_1}, as the following lemma shows.

Lemma 3.9. *Consider a point u_0, where $w_0 > 0$ without loss of generality. For $0 < w_{i_1} < w_0$, the nonclassical 1-shock from w_{i_1} to w_m where $w_m = h(w_{i_1})$ is determined by the kinetic relation (3.19) is always an admissible nonclassical 1-shock. For $w_{i_1} > w_0$, the nonclassical shock from w_{i_1} to w_m, where $w_{i_1} = w_{i_2}$ and $w_{i_2} = g(w_m)$, is determined from the $\mathcal{H}_2$ kinetics, is only admissible if $w_m \in (-w_{i_2} - w_l, -w_{i_2}]$.*

Remark. The function $h(w_{i_1})$ for the symmetric kinetics in the 1-wave family is the inverse of $g(\cdot)$, which yields the right-hand state for nonclassical 2-shocks. Since, as we will show in Theorem 3.10, the function $g(\cdot)$ is monotone in its argument, such an inverse exists and is well defined.

Proof of Lemma 3.9. For $w_{i_1} > 0$, we have $w_m < 0$ and $w_{i_2} > 0$. By Lemma 3.3, $w_{i_2} \in I_2 := (-w_m/2, -w_m]$. For $w_{i_1} < w_0$, which corresponds to the rarefaction/nonclassical 1-shock wave-fan, $w_m \in I_1 = (-2w_{i_1}, -w_{i_1}] = (-2w_{i_2}, -w_{i_2}]$, and therefore $w_{i_2} \in I_2$ iff $w_m \in I_1$.

In the case $w_{i_1} > w_0$, Lemma 3.4 stipulates that there can be a nonclassical shock joining u_{i_1} to u_m, if $w_m \in I_3 = (-w_{i_1} - w_0, -w_{i_1}]$, and by the symmetric kinetics assumption, $I_3 = (-w_{i_2} - w_0, -w_{i_2}]$. Meanwhile, for the nonclassical 2-shock, $w_m \in (-2w_{i_2}, -w_{i_2}]$ which contains the interval I_3, since $w_0 < w_{i_2}$. ∎

Remark. The intervals I_3 and I_4 are almost identical for $w_0 \approx w_{i_1}$. When $w_{i_1} \gg w_0$, however, the interval I_3 becomes an ever-diminishing fraction of I_4, relegated to the upper end containing the "most" nonclassical shocks. As $w_{i_2} \to \infty$, unless the kinetic relation selects $w_{i_2} = -w_m$, no admissible, symmetric 1-shock can be constructed.

We prepare for the construction of the nonclassical wave curves, with a given kinetics, by proving that the 2-shocks selected by (3.19) have $w_+ = g(w_-)$ monotone decreasing in w_-.

THEOREM 3.10. *For a nonclassical* 2*-shock between* (v_-, w_-) *and* (v_+, w_+), *with* $w_+ = g(w_-)$ *selected by the kinetic relation* (3.19),

$$\frac{d\,g(w_-)}{d\,w_-} < 0 \quad \text{for } s \in [m, \infty)\,. \tag{3.28}$$

Proof of Theorem 3.10. In light of the Rankine–Hugoniot condition, we may view the selection of a unique right-hand state, w_+, alternatively as the selection of a unique (nonclassical) shock speed, $s(w_-)$. Thus we may reexpress the kinetic relation (3.19) as

$$\mathcal{D}(w_-; s) = \varphi(s). \tag{3.29}$$

Taking the derivative with respect to w_- in (3.29) gives

$$\frac{\partial \mathcal{D}}{\partial w_-} + \frac{\partial \mathcal{D}}{\partial s}\,\frac{\partial s}{\partial w_-} = \varphi'(s)\,\frac{\partial s}{\partial w_-}. \tag{3.30}$$

Rearrangement of (3.30) leads to

$$\frac{\partial s}{\partial w_-} = \frac{\partial \mathcal{D}/\partial w_-}{\varphi' - \partial \mathcal{D}/\partial s}\,. \tag{3.31}$$

Comparing the functions D and $\mathcal{D}$, we find

$$\frac{\partial D}{\partial w_-} = \frac{\partial \mathcal{D}}{\partial w_-} \quad \text{and} \quad \frac{\partial D}{\partial w_+}\,\frac{\partial w_+}{\partial s} = \frac{\partial \mathcal{D}}{\partial s}. \tag{3.32}$$

For the $\mathcal{H}_2$ nonclassical shocks, we have from Theorem 3.8 that $\partial D/\partial w_+ < 0$ and $\partial w_+/\partial s < 0$, so that by (3.32) we have $\partial \mathcal{D}/\partial s > 0$. In addition, since $\varphi' < 0$ by (3.20), the denominator in (3.31) is always negative. We now use the first equality of (3.32) to compute the sign of the numerator in (3.31).

We regard s as a parameter and compute the derivative of (3.23) with respect to w_-, where we have

$$w_+^2 = -w_-\,w_+ + S - w_-^2 \tag{3.33}$$

from the Rankine–Hugoniot relation; here we have defined $S = s^2 - m^2$. We note that $3\,w_-^2/4 \le S \le w_-^2$. Taking the derivative of (3.33) gives $w'_+ = -(w_+ + 2w_-)/(2w_+ + w_-)$. A straightforward calculation from (3.23) using these quantities results in

$$\frac{\partial D}{\partial w_-} = \frac{-2\,\bar{c}(w_-; w_+)}{2w_+ + w_-}\left[S - \frac{3\,w_-}{2}(w_- + w_+)\right](\,S - 3\,w_-^2\,)\,. \tag{3.34}$$

The first factor is positive, since $\bar{c}(w_-; w_+) > 0$ and $2\,w_+ + w_- < 0$. The second factor in (3.34) is greater than or equal to $S - 3\,w_-^2/4$ and so is also positive. Finally, from the above bounds on S, the third factor in (3.33) is negative. Thus from (3.34) we have $\partial D/\partial w_- < 0$ for Case A. This implies, from (3.31), that $\partial s/\partial w_- > 0$. The result (3.28) then follows from the Rankine–Hugoniot condition. This completes the proof of Theorem 3.10. ∎

COROLLARY 3.11. *The nonclassical* 1*-shock between* (v_{i_1}, w_{i_1}) *and* (v_m, w_m) *with* $w_{i_1} = w_{i_2}$ *from symmetric kinetics, where* $w_{i_2} = g(w_m)$*, has the monotonicity property*

$$\frac{d\,h(w_{i_1})}{d\,w_{i_1}} < 0 \quad \textit{for} \quad s \in (-\infty, -m]. \tag{3.35}$$

We now turn to construction of the nonclassical wave curves, beginning with $\mathcal{W}_1^{nc}(v_0, w_0)$. The point $u_0 = (v_0, w_0)$ is arbitrary, but we take $w_0 > 0$ here for definiteness. Just as in the Liu construction, the wave-curve may be extended indefinitely to the right, along the classical portion of the $\mathcal{H}_1(v_0, w_0)$ curve. Similarly, $\mathcal{W}_1^{nc}(v_0, w_0)$ may be continued to the left until it reaches the point $(v, w) = (\tilde{v}, 0)$, with $\tilde{v}$ given by (3.10), along the integral curve $\mathcal{O}_1(v_0, w_0)$.

To extend this 1-wave curve into the region with $w < 0$, we utilize the symmetric kinetics. For $0 \le w_{i_1} < w_0$, it is possible by Lemma 3.3 and Corollary 3.11 to connect (v_0, w_0) to a point $(v_m, h(w_{i_1}))$, with $-2w_0 < h(w_{i_1}) \le 0$, by an $\mathcal{O}_1$-$\mathcal{H}_1^-$ wave fan. The union of these points, as w_{i_1} varies between w_0 and zero, is given by the curve

$$\begin{aligned} \Gamma_1 = \Big\{ &(v_m, w_m) \in \mathcal{H}_1^-(v_{i_1}, w_{i_1}) \mid w_m = h(w_{i_1}) \in (-2\,w_{i_1}, -w_{i_1}], \\ &w_{i_1} \in \mathcal{O}_1(v_0, w_0),\ \ 0 < w_{i_1} < w_0 \Big\}. \end{aligned} \tag{3.36}$$

By the monotonicity property of $w_m = h(w_{i_1})$, the left-hand endpoint of Γ_1, which represents a single nonclassical shock, must be the point $(v, w) = (v_0^*, h(w_0))$ $\in \mathcal{H}_1^-(v_0, w_0)$, where the value of v_0^* is found from the Hugoniot curve (3.6).

According to Lemma 3.9, when $w_{i_1} > w_0$ there will be a nonclassical $\mathcal{H}_1^+$-$\mathcal{H}_1^-$ wave fan, connecting (v_0, w_0) to $(v_m, h(w_{i_1}))$ iff

$$\psi_h(w_{i_1}; w_0) = h(w_{i_1}) + w_{i_1} + w_0 \ge 0 \tag{3.37}$$

holds, where $h(w_{i_1})$ is the value of w_m selected by the kinetic relation for w_{i_1}. Note that the monotonicity property, from Corollary 3.11, of h does not imply the satisfaction or failure of condition (3.37) and, for a very general kinetic function $\varphi(s)$ in Theorem 3.8, there may be successive intervals in $w_{i_1} > w_0$ where nonclassical shocks are alternately allowed or disallowed.

Since $h(w_{i_1})$ changes smoothly with w_{i_1}, we must have $\psi_h(w_{i_1}; w_0) = 0$ in (3.37) just before it becomes negative, for a slightly larger w_{i_1}. From Lemma 3.4, equality in (3.37) implies that $(v_m, h(w_{i_1}))$ lies on the classical shock curve $\mathcal{H}_1^-(v_0, w_0)$.

We therefore augment the symmetric kinetics for the 1-wave nonclassical shocks by the additional requirement that *if, for a given* w_0*, we have* $\psi_h(w_{i_1}; w_0) < 0$*, at some* $w_m = h(w_{i_1})$*, determined from symmetric kinetics, then the point* $(v_m, w_m) \in \mathcal{W}_1^{nc}(v_0, w_0)$ *is chosen by requiring* $(v_m, w_m) \in \mathcal{H}_1^-(v_0, w_0)$.

By Lemma 3.9, $\psi_h(w_0; w_0) \ge 0$. If there is strict inequality, the nonclassical $\mathcal{H}_1^+$-$\mathcal{H}_1^-$ wave fan will persist until $w_{i_1} = \tilde{w}_1$, where ψ_h switches from positive to negative. Note that $\tilde{w}_1 = w_0$ if $\psi_h(w_0; w_0) = 0$. According to our augmented symmetric kinetics, we continue $\mathcal{W}_1^{nc}(v_0, w_0)$ as a portion of $\mathcal{H}_1^-(v_0, w_0)$ until $w_{i_1} = \tilde{w}_2$, where ψ_h changes from negative to positive. The next segment—to the left of the previous one, as w_m is decreasing with increasing w_{i_1}—of $\mathcal{W}_1^{nc}(v_0, w_0)$ will be a nonclassical one, continuing until $w_{i_1} = \tilde{w}_3$, and so on.

This pattern of alternating classical and nonclassical portions of $\mathcal{W}_1^{nc}(v_0, w_0)$ may continue indefinitely. Regardless of the pattern of classical and nonclassical curves, it follows from Lemmas 3.1 and 3.4 that for (v_m, w_m) on $\mathcal{W}_1^{nc}(v_0, w_0)$, we have $v_m \to -\infty$ as $w_m \to -\infty$.

Let $\{\tilde{w}_k\}, \quad k = 0, 1, 2, \dots$, with $\tilde{w}_0 = w_0 \le \tilde{w}_1 < \tilde{w}_2 < \cdots$, be the set of points where $w_{i_1} \ge w_0$ has $\psi_h(w_{i_1}; w_0) = 0$. From the above argument, $\psi_h(w_{i_1}; w_0) > 0$ for $\tilde{w}_{2k} < w_{i_1} < \tilde{w}_{2k+1}$, while $\psi_h(w_{i_1}; w_0) < 0$ when $\tilde{w}_{2k+1} < w_{i_1} < \tilde{w}_{2k+2}$. We can then describe the portions of allowable nonclassical $\mathcal{H}_1^+$-$\mathcal{H}_1^-$ wave fans by

$$\Gamma_2^{(2k)} = \Big\{ (v_m, w_m) \in \mathcal{H}_1^-(v_{i_1}, w_{i_1}) \mid w_m = h(w_{i_1}), (v_{i_1}, w_{i_1}) \in \mathcal{H}_1^+(v_0, w_0), \\ \tilde{w}_{2k} \le w_{i_1} \le \tilde{w}_{2k+1} \Big\} \tag{3.38}$$

for $k = 0, 1, 2, \dots$. The right-hand endpoint of $\Gamma_2^{(0)}$ represents a single nonclassical shock from (v_0, w_0) to the point $(v_0^*, h(w_0))$, so that this precisely matches the left-hand endpoint of the curve Γ_1, calculated previously. The curve $\mathcal{W}_1^{nc}(v_0, w_0)$ is therefore continuous at $w_m = h(w_0)$. The left-hand endpoint of $\Gamma_2^{(0)}$, as well as both endpoints of $\Gamma_2^{(2k)}$ for $k > 0$, join continuously to the (classical) Hugoniot curve $\mathcal{H}_1^-(v_0, w_0)$, according to Lemma 3.4. We denote the segments of $\mathcal{H}_1^-(v_0, w_0)$, used in this construction, by

$$\Gamma_2^{(2k+1)} = \Big\{ (v_m, w_m) \in \mathcal{H}_1^-(v_0, w_0), \mid w_m = h(w_{i_1}), (v_{i_1}, w_{i_1}) \in \mathcal{H}_1^+(v_0, w_0), \\ \tilde{w}_{2k+1} \le w_{i_1} \le \tilde{w}_{2k+2} \Big\} \tag{3.39}$$

for $k = 0, 1, 2, \dots$. The curve $\mathcal{W}_1^{nc}(v_0, w_0)$ is then given by

$$\mathcal{W}_1^{nc}(v_0, w_0) = \begin{cases} \mathcal{H}_1^+(v_0, w_0) & \text{for } w > w_0, \\ \mathcal{O}_1(v_0, w_0) & \text{for } 0 \le w \le w_0, \\ \Gamma_1 & \text{for } h(w_0) \le w < 0, \\ \Gamma_2^{(0)} \cup \Gamma_2^{(1)} \cup \Gamma_2^{(2)} \cup \cdots & \text{for } w < h(w_0), \end{cases} \tag{3.40}$$

where $h(w_0) < 0$ is determined by symmetric kinetics, and w_0 is taken to be positive. Together, the above union of curves stretches continuously from $(v, w) = (-\infty, -\infty)$ to $(v, w) = (\infty, \infty)$.

We complete the discussion of $\mathcal{W}_1^{nc}(v_0, w_0)$ by showing that it increases monotonically in v as a function of w. Since the classical portions of this curve are known from Lemma 3.1 to be monotone increasing in w, it remains to show that the nonclassical segments are also increasing. The next lemma shows that, in fact, the curves Γ_1 and $\Gamma_2^{(2k)}$ are monotone increasing.

LEMMA 3.12. *Suppose $(v_m, w_m) \in \Gamma_1$ or $(v_m, w_m) \in \Gamma_2^{(2k)}$. Then v_m is monotonically increasing with w_m.*

Proof of Lemma 3.12. For the point (w_m, v_m) on Γ_1, one calculates that

$$\frac{dv_m}{dw_{i_1}} = -\frac{\left(\bar{c}(w_{i_1}; w_m) - \bar{c}(w_{i_1})\right)^2}{2\,\bar{c}(w_{i_1}; w_m)} + \frac{dw_m}{dw_{i_1}} \left(c(w_{i_1}; w_m) + \frac{(w_m - w_{i_1})(2\,w_m + w_{i_1})}{2\,c(w_{i_1}; w_m)} \right).$$

The first term is nonpositive, while the coefficient of the dw_m/dw_{i_1} can be shown to be positive. Since we have $dh(w_{i_1})/dw_{i_1} < 0$, from Corollary 3.11, it follows that $dv_m/dw_{i_1} < 0$, and so Γ_1 increases from left to right in w. It can further be shown that for $(v_m, w_m) \in \Gamma_2^{(2k)}$,

$$\frac{dv_m}{dw_{i_1}} = (\bar{c}(w_0; w_{i_1}) - \bar{c}(w_m; w_{i_1}))\left(1 - \frac{\bar{c}^2(w_{i_1})}{\bar{c}(w_0; w_{i_1}) - \bar{c}(w_m; w_{i_1})}\right) + \frac{dw_m}{dv_{i_1}}\left(\bar{c}(w_{i_1}; w_m) + \frac{(2\,w_m + w_{i_1})(w_m - w_{i_1})}{2\,\bar{c}(w_{i_1}; w_m)}\right).$$

Since we have the inequalities $\bar{c}(w_{i_1}) \geq \bar{c}(w_0; w_{i_1}) \geq \bar{c}(w_{i_1}; w_m) \geq 0$, the first term is negative, while the coefficient of dw_m/dv_{i_1} is again positive. Applying Corollary 3.11, regarding the sign of $dh(w_{i_1})/dw_{i_1}$, yields the desired monotonicity for v_m as a function of w_m. ∎

We now turn to the construction of $\mathcal{W}_2^{nc}(v_m, w_m)$. For this discussion, $u_m = (v_m, w_m)$ is taken to be arbitrary. Alternatively, we can view this point as $u_m \in \mathcal{W}_1^{nc}(v_0, w_0)$ for some $u_0 = (v_0, w_0)$. To be definite, we take $w_m > 0$, but this discussion could be extended to $w_m < 0$ with little complication. We are interested in the set of points $u_1 = (v_1, w_1)$ that can be connected to u_m through either a rarefaction, classical shock, shock-rarefaction, or a pair of shocks with positive wave speeds; in the latter two cases, there will be an intermediate state, $u_{i_2} = (v_{i_2}, w_{i_2})$, between u_m and u_1. A specific kinetic function has been imposed, so that the kinetic relation (3.19) selects a unique value $w_{i_2} = g(w_m)$ from among the possible nonclassical shocks in $\mathcal{H}_2(v_m, w_m)$.

As in the classical case (see Lemma 3.2), when $w > w_m$, this portion of $\mathcal{W}_2^{nc}(v_m, w_m)$ is just $\mathcal{O}_2(v_m, w_m)$. Similarly, when $0 < w < w_m$, we have that this section of $\mathcal{W}_2^{nc}(v_m, w_m)$ matches the classical shock curve $\mathcal{H}_2(v_m, w_m)$. To determine how far this classical shock curve penetrates into the region $w < 0$, however, the specific kinetics must be taken into account, as we do below.

The point $u_{i_2} \in \mathcal{H}_2(v_m, w_m)$, with $-w_m \leq w_{i_2} < -w_m/2$, is the unique right-hand state for the nonclassical shock, determined by the kinetic relation (3.19). From u_{i_2}, the solution may be continued, according to Lemma 3.6, along $\mathcal{O}_2(v_{i_2}, w_{i_2})$ for $w < w_{i_2}$. We denote this portion of $\mathcal{W}_2^{nc}(v_m, w_m)$ by

$$\Gamma_4 = \left\{(v, w) \in \mathcal{O}_2(v_{i_2}, w_{i_2}) \mid w_{i_2} = g(w_m), \quad -\infty < w \leq w_{i_2}\right\}. \tag{3.41}$$

According to Lemma 3.6, one may also continue from u_{i_2} to $u_1 = (v_1, w_1) \in \mathcal{H}_2(v_{i_2}, w_{i_2})$ for $w_{i_2} \leq w_1 \leq -w_m - w_{i_2}$. This portion of $\mathcal{W}_2^{nc}(v_m, w_m)$ will be labeled by

$$\Gamma_3 = \left\{(v, w) \in \mathcal{H}_2(v_{i_2}, w_{i_2}) \mid w_{i_2} = g(w_m), \quad w_{i_2} \leq w \leq -w_m - w_{i_2}\right\}. \tag{3.42}$$

For $u_1 \in \mathcal{H}_2(v_{i_2}, w_{i_2})$, with $w_1 = -w_m - w_{i_2}$, we saw in Lemma 3.5 that $u_1 \in \mathcal{H}_2(v_m, w_m)$ as well. We may therefore complete the construction of $\mathcal{W}_2^{nc}(v_m, w_m)$ in a continuous manner by extending the classical portion of $\mathcal{H}_2(v_m, w_m)$ until $w = -w_m - w_{i_2}$. This continuous, nonclassical 2-wave curve is then given by

$$\mathcal{W}_2^{nc}(v_m, w_m) = \begin{cases} \mathcal{O}_2(v_m, w_m) & \text{for } w > w_m, \\ \mathcal{H}_2(v_m, w_m) & \text{for } -w_m - w_{i_2} < w \leq w_m, \\ \Gamma_3 & \text{for } w_{i_2} \leq w \leq -w_m - w_{i_2}, \\ \Gamma_4 & \text{for } w < w_{i_2}, \end{cases} \tag{3.43}$$

where $w_{i_2} = g(w_m)$ comes from the kinetic relation (3.19). We now show that the curve $\mathcal{W}_2^{nc}(u_m)$ of (3.43) has v monotone decreasing with w.

LEMMA 3.13. *The curve $\mathcal{W}_2^{nc}(v_m, w_m)$ defined in* (3.43) *is continuous, with v monotone decreasing in w, from $(v, w) = (-\infty, \infty)$ to $(v, w) = (\infty, -\infty)$. Furthermore, $\mathcal{W}_2^{nc}(v_m, w_m)$ is $\mathcal{C}^{\in}$ except at $w = -w_m - w_{i_2}$, where it is merely continuous.*

Proof of Lemma 3.13. The monotonicity of $\mathcal{W}_2^{nc}(v_m, w_m)$ follows from it being the continuous union of four monotone decreasing curves: Γ_4, which is a portion of $\mathcal{O}_2(v_{i_2}, w_{i_2})$, has v decreasing for increasing w by (3.9). This integral curve naturally joins $\mathcal{H}_2(v_{i_2}, w_{i_2})$ at u_{i_2} with second-order contact, so that Γ_3 and Γ_4, together, form a $\mathcal{C}^{\in}$ curve. By (3.7), we have Γ_3 decreasing as w increases.

From Lemma 3.6, Γ_3 and $\mathcal{H}_2(v_m, w_m)$ meet at $w = -w_m - w_{i_2}$, implying continuity. The remaining portion of $\mathcal{W}_2^{nc}(v_m, w_m)$ is classical, and its continuity and monotonicity follow from Lemma 3.2.

The infinite extent, in v, of $\mathcal{W}_2^{nc}(v_m, w_m)$ follows from the divergence of the integral in (3.9), as $|w| \to \infty$. This proves Lemma 3.13. $\square$

Combining Lemmas 3.12 and 3.13, regarding the infinite extent, continuity, and the respective monotonicities of the nonclassical wave curves, $\mathcal{W}_1^{nc}(v_0, w_0)$ and $\mathcal{W}_2^{nc}(v_m, w_m)$, we have in the following theorem our main result of this section.

THEOREM 3.14. *Given a point (v_0, w_0), the Riemann problem for system* (3.1) *with initial data (u_0, u_1), where $u_1 = (v_1, w_1)$ is an arbitrary point, has a unique solution in the class of nonclassical shocks, given a kinetic function $\varphi(s)$ satisfying assumptions* (3.20)–(3.22), *and assuming augmented symmetric kinetics for the* 1*-wave family.*

4. Nonclassical shocks in elastodynamics (2).

4.1. Convergence result. For the model of section 3, the convergence of some approximations toward weak solutions is easily established, applying the method of compensated compactness (Murat [49], Tartar [60], DiPerna [13]) as we show in this subsection. With no uniqueness result available for nonclassical solutions, only subsequences of solutions can be shown to converge. It is one of the challenging open problems in this area to extend the kinetic relation, introduced in this paper for traveling waves, to more general solutions. This is because the kinetic relation has been introduced for functions of bounded variation, while the compensated compactness approach provides solutions in a functional space of less regular functions (i.e., L^p).

Consider the augmented version of the elastodynamics system:

$$\begin{aligned} \partial_t v - \partial_x \sigma(w) &= \epsilon\, \partial_{xx} v - \alpha\, \epsilon^2\, \partial_{xxx} w, \\ \partial_t w - \partial_x v &= 0, \end{aligned} \tag{4.1}$$

where ϵ and α are positive constants. Here σ is given by (3.2) as in section 3. Regularization terms as in the right-hand side of (4.1) were first studied by Slemrod (see [59] and Fan and Slemrod [15]) for the case that σ is decreasing in some interval, which models phase transitions in materials or in fluids; therein the dispersion term models the capillarity effect of the fluid. As we can demonstrate numerically, the sign of the dispersion term in (4.1) corresponds to that where nonclassical behavior is observed.

As the coefficients in front of the diffusion and dispersion terms vanish, the solutions to (4.1) converge to a nonclassical solution to the hyperbolic model (3.1). Observe that the presence of the dispersion term in the right-hand side of the first equation in (4.1) (and the absence of diffusion in the second equation) prevents obtaining an L^∞ bound by the theory of invariant regions à la Chuey, Conley, and Smoller [7]. The theorem below uses L^p estimates, instead.

Define the internal energy W by $W'(w) = \sigma(w)$. From (3.2) one gets $W(w) = \big(w^4 + 2\, m^2\, w^2\big)/4$.

THEOREM 4.1. (1) *Let* $\big(v^\epsilon, w^\epsilon\big)$, *with* $\alpha \geq 0$ *fixed, be a family of solutions to* (4.1) *assuming at* $t = 0$ *a Cauchy data* $\big(v_0^\epsilon, w_0^\epsilon\big)$ *satisfying uniform bounds in* ϵ *in the following spaces:*

$$v_0^\epsilon \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}), \qquad w_0^\epsilon \in L^1(\mathbb{R}) \cap L^4(\mathbb{R}), \qquad \epsilon^{1/2}\, \partial_x w_0^\epsilon \in L^2(\mathbb{R}). \tag{4.2}$$

Then the sequences v^ϵ *and* w^ϵ *remain uniformly bounded in* $L^\infty\big(\mathbb{R}_+, L^2(\mathbb{R})\big)$ *and* $L^\infty\big(\mathbb{R}_+, L^4(\mathbb{R})\big)$, *respectively, and converge almost everywhere to limiting functions* v *and* w, *solutions to the hyperbolic system* (3.1).

(2) *The entropy pair* $(U, F) = \big(v^2/2 + W(w), -v\,\sigma(w)\big)$ *is compatible in the sense* (2.3) *with the diffusive-dispersive regularization* (4.1). *Limits of traveling wave solutions to* (4.1), *additionally, satisfy the entropy inequality*

$$\partial_t \left(\frac{v^2}{2} + W(w) \right) - \partial_x\big(v\,\sigma(w)\big) \leq 0. \tag{4.3}$$

We do not expect the entropy inequalities

$$\partial_t U(v, w) + \partial_x F(v, w) \leq 0, \tag{4.4}$$

with $U(v, w) \neq v^2/2 + W(w)$ (up to a linear function of v and w), to hold in general.

Proof of Theorem 4.1. The bounds in L^2 and L^4 follow from the following standard energy estimate. Multiplying the first equation in (4.1) by σ and the second one by v, we arrive at

$$\partial_t\big(W(w) + v^2/2\big) - \partial_x\big(v\,\sigma(w)\big) = -\epsilon\,|\partial_x v|^2 + \epsilon\,\partial_x\big(v\,\partial_x v\big) - \alpha\,\epsilon^2\,\partial_x\big(v\,\partial_{xx} w\big) + \alpha\,\epsilon^2 \partial_x v\,\partial_{xx} w.$$

Using the second equation in (4.1), we rewrite $\partial_x v\,\partial_{xx} w = \partial_t w\,\partial_{xx} w = \partial_x\big(\partial_t w\,\partial_x w\big) - \partial_t\big(|\partial_x w|^2/2\big)$. Therefore we obtain the following entropy balance:

$$\begin{aligned} &\partial_t\big(W(w) + v^2/2 + \alpha\,\epsilon^2\,|\partial_x w|^2/2\big) - \partial_x\big(v\,\sigma(w)\big) \\ &\quad = -\epsilon\,|\partial_x v|^2 + \epsilon\,\partial_{xx}\big(v^2/2\big) - \alpha\,\epsilon^2\,\partial_x\big(v\,\partial_{xx} w\big) + \alpha\,\epsilon^2 \partial_x\big(\partial_x v\,\partial_x w\big). \end{aligned} \tag{4.5}$$

This leads to the following uniform bound:

$$\begin{aligned} &\int_{\mathbb{R}} \big(W(w) + v^2/2 + \alpha\,\epsilon^2\,|\partial_x w|^2/2\big)(T)\,dx + \int_0^T \int_{\mathbb{R}} \epsilon\,|\partial_x v|^2\,dxdt \\ &= \int_{\mathbb{R}} \big(W(w) + v^2/2 + \alpha\,\epsilon^2\,|\partial_x w|^2/2\big)(0)\,dx \leq O(1), \end{aligned} \tag{4.6}$$

where we have used (4.2) and where $O(1)$ denotes a constant independent on ϵ.

Multiply the first equation in (4.1) by $\partial_x w$ and integrate in space and time to write, on one hand,

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}} \big(\partial_x w\,\partial_t v - \partial_x w\,\sigma_w(w)\partial_x w\big)\,dxdt \\ &= \left[\int_{\mathbb{R}} \partial_x w\,v\,dx\right]_0^T - \int_0^T \int_{\mathbb{R}} \partial_{xx} v\,v\,dxdt - \int_0^T \int_{\mathbb{R}} \sigma_w(w)\,|\partial_x w|^2\,dxdt \\ &= \int_{\mathbb{R}} \partial_x w(T)\,v(T)\,dx - \int_{\mathbb{R}} \partial_x w(0)\,v(0)\,dx + \int_0^T \int_{\mathbb{R}} |\partial_x v|^2\,dxdt \\ &\quad - \int_0^T \int_{\mathbb{R}} \sigma_w(w)\,|\partial_x w|^2\,dxdt \end{aligned}$$

and, on the other hand,

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}} \partial_x w \left(\epsilon\, \partial_{xx} v - \alpha\, \epsilon^2\, \partial_{xxx} w\right) dxdt\\&= \int_0^T \int_{\mathbb{R}} \epsilon\, \partial_x w\, \partial_{tx} w\, dxdt + \alpha\, \epsilon^2 \int_0^T \int_{\mathbb{R}} |\partial_{xx} w|^2\, dxdt\\&= \left[\epsilon \int_{\mathbb{R}} |\partial_x w|^2/2\, dx\right]_0^T + \alpha\, \epsilon^2 \int_0^T \int_{\mathbb{R}} |\partial_{xx} w|^2\, dxdt.\end{aligned}$$

Observe that

$$\left| \int_{\mathbb{R}} \partial_x w(T)\, v(T)\, dx\right| \le \epsilon \int_{\mathbb{R}} |\partial_x w(T)|^2/2\, dx + \left(2\epsilon\right)^{-1} \int_{\mathbb{R}} |v(T)|^2\, dx,$$

and similarly for the term $\partial_x w(0)\, v(0)$. Finally, combining the above formulas, we obtain

(4.7)
$$\begin{aligned}&\int_0^T \int_{\mathbb{R}} \epsilon\, \sigma_w(w)\, |\partial_x w|^2\, dxdt + \alpha\, \epsilon^2 \int_0^T \int_{\mathbb{R}} |\partial_{xx} w|^2\, dxdt\\&\qquad \le \int_0^T \int_{\mathbb{R}} \epsilon\, |\partial_x v|^2\, dxdt + \epsilon^2 \int_{\mathbb{R}} |\partial_x w(0)|^2\, dx + \int_{\mathbb{R}} |v(T)|^2/2\, dx, + \int_{\mathbb{R}} |v(0)|^2/2\, dx.\end{aligned}$$

Combining (4.6) and (4.7) and using the form (3.2) of the function σ, we obtain the uniform bounds

$$\int_{\mathbb{R}} \left(v(T)^2 + w(T)^2 + w(T)^4\right) dx + \int_{\mathbb{R}} \alpha\, \epsilon\, |\partial_x w(T)|^2\, dx \le O(1), \tag{4.8}$$

$$\int_0^T \int_{\mathbb{R}} \left(\epsilon\, |\partial_x v|^2 + \epsilon\, |\partial_x w|^2 + \alpha\, \epsilon^2\, |\partial_{xx} w|^2\right) dxdt \le O(1). \tag{4.9}$$

Using the $L^2 \times L^4$ bound derived for the sequence (v_ϵ, w_ϵ), we introduce a Young measure representing possible oscillations in the sequence as $\epsilon \to 0$. The estimates (4.8)–(4.9) are the basis for applying DiPerna's argument in [13], which shows that the Young measure satisfies the so-called Tartar commutation equation. The standard reduction theorem, stating that it must reduce to a Dirac mass, does not apply here since the support of the Young measure is not bounded.

Instead, the work by Shearer [55] and Serre and Shearer [54] based on L^p estimates does apply. The system (3.1) is strictly hyperbolic and the constitutive equation σ possesses a single inflection point. The theorem in [54] implies that there exists a limiting function $(v, w) \in L^\infty(L^2 \times L^4)$ such that the sequence strongly converges to (v, w) in the sense

$$\begin{aligned}&v_\epsilon \to v \text{ in } L^p \text{ for all } p < 2,\\&w_\epsilon \to w \text{ in } L^p \text{ for all } p < 2.\end{aligned} \tag{4.10}$$

Observe that (4.10) suffices for the passage to the limit in (4.1) and in order to derive (3.1): the nonlinearity $\sigma(w)$ is cubic while we have a control of w in L^4 by the entropy estimate (4.6).

Showing that the natural entropy of the system (3.1) is compatible with the regularization (4.1) is easy from (2.3). It is a classical matter (see Schonbek [53] and, also, Hayes and LeFloch [22] for the analogous case of scalar equations with vanishing diffusion and dispersion) to check that, in view of (4.8)–(4.9), the right-hand side of (4.5) converges to zero in the sense of distributions. The entropy flux does converge to its corresponding limit since $\sigma(w^\epsilon)$ converges strongly to $\sigma(w)$. The term $\alpha\,\epsilon\,\partial_t|\partial_x w_\epsilon|^2$ converges to zero in the sense of distributions thanks to (4.8)–(4.9). Let us, equivalently, check that the product of v and $\alpha\,\epsilon^2\,\partial_{xxx}w$ converges to zero. Namely, for each smooth function θ with compact support,

$$\begin{aligned}
&\Big|\int_0^T\int_{\mathbb{R}}\epsilon^2\,v\,\partial_{xxx}w\,\theta\,dxdt\Big|\\
&\le\Big|\int_0^T\int_{\mathbb{R}}\epsilon^2\,\partial_x v\,\partial_{xx}w\,\theta\,dxdt\Big|+\Big|\int_0^T\int_{\mathbb{R}}\epsilon^2\,v\,\partial_{xx}w\,\partial_x\theta\,dxdt\Big|\\
&\le O(1)\,\epsilon^2\,\|\partial_x v\|_{L^2((0,T)\times\mathbb{R})}\,\|\partial_{xx}w\|_{L^2((0,T)\times\mathbb{R})}+O(1)\,\epsilon^2\,\|v\|_{L^2((0,T)\times\mathbb{R})}\,\|\partial_{xx}w\|_{L^2((0,T)\times\mathbb{R})}\\
&\le O(1)\,\epsilon^{1/2}+O(1)\,\epsilon\;\to\;0.
\end{aligned}$$

Further estimates are needed to treat the entropy term in general, since we know only that $W(w^\epsilon)$ is bounded in $L^\infty_t(L^1_x)$ and, therefore, could a priori converge to a bounded Radon measure. However, in the special case of (smooth) traveling wave solutions to (4.1), it is straightforward to deduce (4.3) follows from the entropy balance (4.5), since all of the terms in the right-hand side of (4.5) have a conservative form but one which is nonpositive. □

We now comment upon the derivation a kinetic relation for (3.1) associated with the regularization (4.1). After rescaling by ϵ, a traveling wave solution (v,w) to (4.1), connecting (v_0,w_0) to (v_1,w_1) and having the speed s, satisfies the following third order system of ODEs:

$$\begin{aligned}
s\,w'+v' &= 0,\\
s\,v'+\sigma(w)' &= -v''+\alpha\,w'''
\end{aligned}$$

together with the conditions $v(\xi)\to v_0$, $w(\xi)\to w_0$ at $\xi\to-\infty$, and $v(\xi)\to v_1$, $w(\xi)\to w_1$ at $\xi\to+\infty$. We also assume that w', w'', and w''' vanish at $\pm\infty$. Eliminating the variable v, we obtain an equation for the scalar-valued function w:

$$-s^2\,w'+\sigma(w)' = s\,w''+\alpha\,w'''.$$

Integrating once, we obtain

$$-s^2\big(w-w_0\big)+\sigma(w)-\sigma(w_0) = s\,w'+\alpha\,w''. \tag{4.11}$$

Given a value for the shock speed s, there exist up to three states that solve the equation giving the equilibrium points of (4.11), i.e.,

$$-s^2\big(w-w_0\big)+\sigma(w)-\sigma(w_0) = 0. \tag{4.12}$$

Namely, these are w_0 itself and (at most) two additional points w_1 and w_2. Since the cubic $\sigma(w)=w^3+m^2w$ has no quadratic term, one must have $w_0+w_1+w_2=0$. Consider the case that w_0 is chosen such that $w_0>0$ and $w_1<w_2<0$ which holds in a certain range of values for s.

Consider for instance waves of the second characteristic family propagating with $s > 0$. From the Liu criterion, it follows that a traveling wave connecting w_0 to w_1 represents a classical shock, while a connection from w_0 to w_2 is a nonclassical shock. A typical feature of (4.11) is the following one [62]: there exists a critical value for the slope $s^\sharp$ such that a traveling wave trajectory connecting to w_1 exists for speeds $s > s^\sharp$ and there exists a connection to w_2 when $s = s^\sharp$.

We emphasize that, given w_0, there exist a unique state w_2 and a unique speed such that w_0 and w_2 can be connected by a nonclassical shock. The traveling wave analysis therefore allows us to write, say,

$$w_2 = g(w_0) \qquad \text{and} \qquad s = s(w_0). \tag{4.13}$$

Using (4.13), the entropy dissipation of the nonclassical shocks can be computed as a function of the left state of the shock. This determines the kinetic function ϕ:

$$\phi(w_0) := D(w_0, w_2) = D\big(w_0, g(w_0)\big). \tag{4.14}$$

Provided the relation $s = s(w_0)$ is one-to-one, one can rewrite (4.14) and obtain the kinetic function expressed as a function of the propagation speed, that is,

$$\varphi(s) := \phi(w_0) \qquad \text{with } s^2 = w_0^2 + g(w_0)^2 + w_0\, g(w_0) + m^2. \tag{4.15}$$

The possibility of writing the kinetic function as a function of a single variable (here w), and hence as a function of the speed s, is a special property of the system (3.1) *and* the regularization (4.1). Other regularizations to (3.1), for which a scalar equation like (4.11) could not be derived, may require a kinetic function of the general form $\phi(v_0, w_0)$.

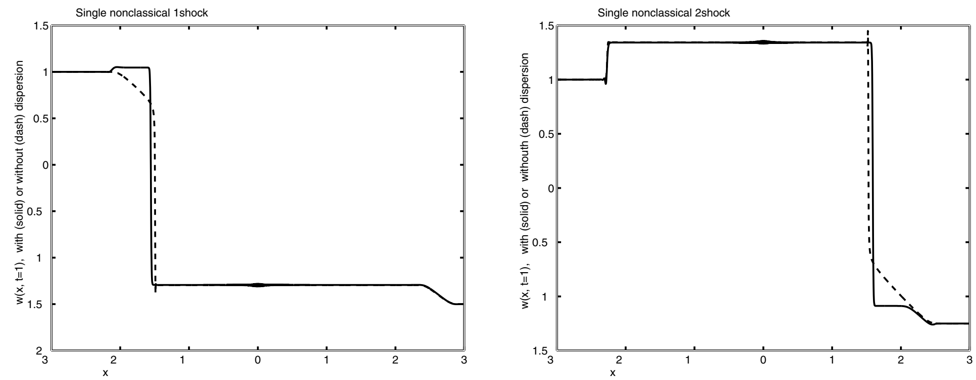
4.2. Numerical experiments. The paper [23] is devoted to the numerical analysis of nonclassical shocks in finite difference schemes. Our purpose here is to illustrate that nonclassical shocks do indeed appear.

In this subsection, we solve the Riemann problem numerically and confirm some of the results enumerated in section 3. We employ the following semidiscrete approximation to the augmented system

$$\begin{aligned} \frac{dv_k}{dt} - \frac{1}{2\,\Delta}\big(\sigma(w_{k+1}) - \sigma(w_{k-1})\big) &= \frac{\epsilon}{\Delta^2}\big(v_{k+1} - 2\,v_k + v_{k-1}\big) \\ &\quad - \frac{\alpha\,\epsilon^2}{2\,\Delta^3}\big(w_{k+2} - 2\,w_{k+1} + 2\,w_{k-1} - w_{k-2}\big), \\ \frac{dw_k}{dt} - \frac{1}{2\,\Delta}\big(v_{k+1} - v_{k-1}\big) &= 0 \end{aligned} \tag{4.16}$$

for functions $w_k(t)$ and $v_k(t)$, where Δ denotes the spatial mesh-size. We integrate this system of ODEs using a fourth-order Runge–Kutta explicit scheme, taking as large a time-step τ as possible. We define $\lambda = \tau/\Delta$. Here we are interested in the continuous model (4.16) for small ϵ. (See [23] for results on numerical schemes.) The following figures may be taken to represent features of the continuous model (4.1): we carefully checked that reducing the mesh size further virtually does not change the numerical results.

The Riemann initial data for the numerical scheme is implemented as $(v_k(0), w_k(0)) = (v_l, w_l)$ for $k \le 0$ and (v_r, w_r) for $k > 0$. In Figures 4.1–4.2, we plot the numerical solution for several choices of initial data and parameters ϵ and α. From these figures,



(a) (b)

FIG. 4.1. *Single nonclassical shock:* (a) *nonclassical* 1-*shock,* (b) *nonclassical* 2-*shock.*

we may compare the classical and nonclassical solutions. All the tests are performed on the interval $x \in [-3, 3]$ and with $m = 1$ in (3.2).

In Figure 4.1(a), we use the initial data $(v_l, w_l) = (1, 1)$ and $(v_r, w_r) = (-1.5, -2)$. The parameters are chosen to be $\Delta = 1/400$, $\lambda = .2$. The component w of the numerical solution is represented in Figure 4.1(a): the dashed line and the solid line correspond to $\alpha = 0$ and $\alpha = 10$, respectively. In the second case we do observe nonclassical behavior, i.e., a nonclassical 1-shock.

Figure 4.1(b) is similar to Figure 4.1(a), except that $(v_r, w_r) = (-1.25, 6)$. The dashed line represents a nonclassical 2-shock.

Figure 4.2 shows an example of a solution containing two nonclassical shocks, a 1-shock propagating in the left direction and a 2-shock going to the right. This is obtained with a suitable choice of the right state: $(v_r, w_r) = (.9, -5)$. The other parameters are the same as before. Figures 4.2(a) and 4.2(b) show the w- and v-component of the numerical solution, respectively.

5. Nonclassical shocks in magnetohydrodynamics.

5.1. Preliminaries. This section deals with a system, introduced by Freistühler [16], arising in the modeling of small amplitude solutions to conservation laws that are rotationally invariant:

$$\begin{aligned} \partial_t v + \partial_x\big((v^2 + w^2)\, v\big) &= 0, \\ \partial_t w + \partial_x\big((v^2 + w^2)\, w\big) &= 0. \end{aligned} \tag{5.1}$$

In magnetohydrodynamics, (v, w) represents transverse components of the magnetic field. This model is relevant to explain certain features observed in the solar wind around the Earth: Cohen and Kulsrud [8] and Wu and Kennel [65]. The model and its variants also arise in nonlinear elasticity. See also the interesting paper by Brio and Hunter [4]. The study of MHD traveling waves has a long history in the mathematical literature (consult, for instance, Conley and Smoller [9]). The system (5.1) has attracted attention of many researchers in recent years: Chen [6], Freistühler [17], Keyfitz and Kranzer [32], Liu and Wang [44], etc. Freistühler and Liu [19] established the nonlinear stability of overcompressive shocks for a parabolic regularization of the system (5.1).

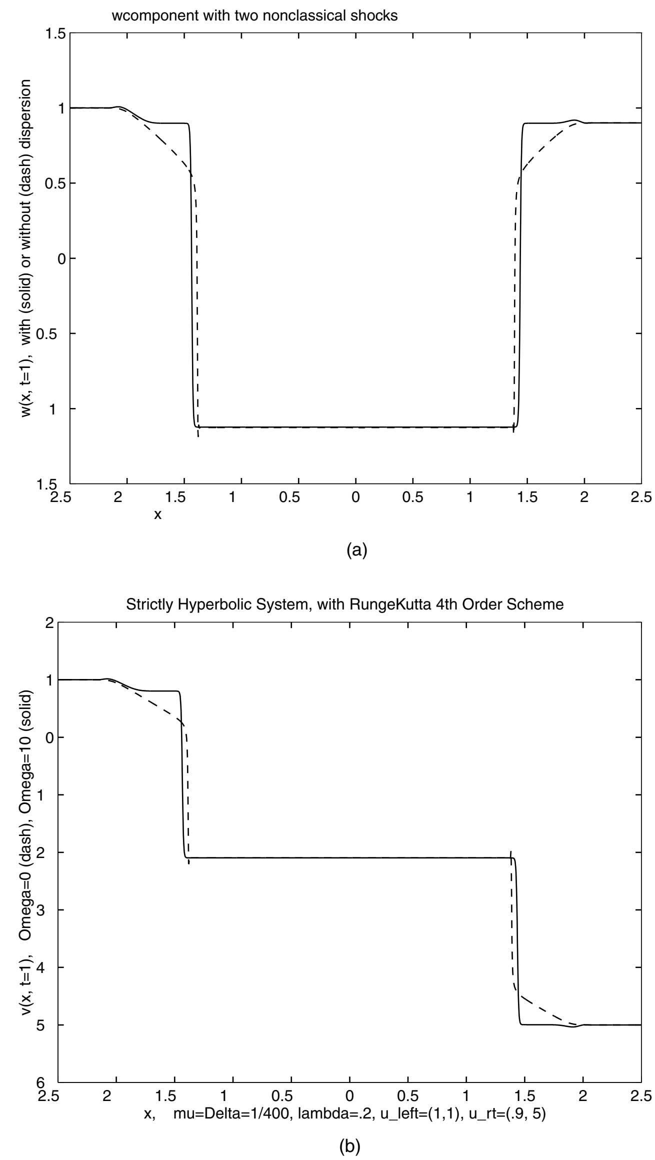


FIG. 4.2. *Nonclassical shocks in both characteristic families:* (a) *w-component,* (b) *v-component.*

For smooth solutions, one can use polar coordinates

$$v = r\cos\theta, \qquad w = r\sin\theta, \qquad r \geq 0, \qquad \theta \in [0, 2\pi), \tag{5.2}$$

and rewrite (5.1) as

$$\partial_t r + \partial_x r^3 = 0, \tag{5.3}$$

$$\partial_t \theta + r^2\, \partial_x \theta = 0. \tag{5.4}$$

The equation (5.3) is a scalar conservation law with a nonconvex (cubic) flux. We deduce from (5.3) that $\lambda_2 = 3\,r^2$ is a wave speed for (5.1); it is the fast mode of the system and the corresponding characteristic field therefore fails to be genuine nonlinear. On the other hand, (5.4) is linearly degenerate since the slow mode wave speed $\lambda_1 = r^2$ is independent of θ.

Observe that the system is strictly hyperbolic everywhere but at the so-called *umbilic point* $v = w = 0$ or equivalently $r = 0$. The change of variable (5.2) is in fact ill defined at $r = 0$ since the angle θ may be arbitrary. The structure (5.3)–(5.4) reflects the property of invariance by rotation or isotropy of (5.1). There exists two main wave families:

• the *rotational discontinuities* keep the radius r constant while the angle θ may vary arbitrarily. Any entropy inequality would be satisfied by rotational discontinuities.

• the *fast shocks* keep the angle θ constant *modulo* π while the radius r may vary arbitrarily. An entropy inequality would select admissible fast shocks among all possible such shocks. Note that a rotational discontinuity always precedes a fast shock.

Consider now particular solutions to (5.1) such that $w = \rho\, v$, where ρ is a given constant. Such solutions will be called *coplanar* in this section. Then both equations in (5.1) reduce to the same equation,

$$\partial_t v + \left(1 + \rho^2\right) \partial_x v^3 \,=\, 0, \tag{5.5}$$

which is a scalar conservation law with cubic flux. Therefore, when the initial data for (5.1) are coplanar, one can attempt to solve the system (5.1) by solving the reduced equation (5.5). This is a saddle issue: the transformation $w = \rho\, v$ need not be compatible with a given regularization added to the right-hand side of (5.1). However, in several instances the solutions to (5.5) turn out to be relevant to describe the solutions to (5.1). Note finally that the "natural" entropy for (5.1),

$$U(v,w) = \frac{1}{2}\left(v^2 + w^2\right) = \frac{r^2}{2}, \qquad F(v,w) = \frac{3}{4}\left(v^2 + w^2\right)^2 = \frac{3\, r^4}{4}, \tag{5.6}$$

reduces, when $w = \rho\, v$, to an entropy pair for (5.5),

$$U(v) = \frac{1}{2} v^2, \qquad F(v,w) = \frac{3}{4}\left(1 + \rho^2\right) v^4, \tag{5.7}$$

which happens to be the one used in [22].

5.2. Unique admissible nonclassical entropy solution. The existence and properties of the nonclassical shocks for the cubic conservation law (5.3) were investigated in Hayes and LeFloch [22]. The equation (5.3), however, is supplemented with the constraint that $r \geq 0$, which prevents us from truly solving (5.3) independently of (5.4) for θ, even for coplanar initial data. The definitions in section 2 extend easily to (5.1) even though the system is not strictly hyperbolic. We are interested in solutions satisfying the single entropy inequality

$$\frac{1}{2}\partial_t\left(v^2 + w^2\right) + \frac{3}{4}\partial_x\left(v^2 + w^2\right)^2 \leq 0. \tag{5.8}$$

Our aim is to investigate the uniqueness of the nonclassical solutions for the system (5.1). Relying on the analysis in [22] we state, without proof, the following result.

THEOREM 5.1. *Consider the Riemann problem for the system* (5.1) *with initial data* (v_l, w_l) *and* (v_r, w_r). *When the data are noncoplanar, then there exists a unique* *solution to the Riemann problem satisfying the entropy inequality* (5.8): *it contains a rotational discontinuity connecting* (v_l, w_l) *to a point* (v_*, w_*) *with* $v_l^2 + w_l^2 = v_*^2 + w_*^2$ *followed by either a fast shock or a rarefaction connecting to* (v_r, w_r).

When the data are coplanar and the angles θ_l *and* θ_r *associated with the initial data satisfy* $\theta_r = \theta_r$ *(mod.* π*), the Riemann problem has a unique solution containing either a classical shock or a rarefaction.*

When the data are coplanar and $\theta_r = \pi + \theta_r$ *(mod.* π*), the Riemann problem admits a one-parameter family of entropy solutions containing a nonclassical shock connecting* (v_l, w_l) *to a point* (v_*, w_*) *with*

$$v_l^2 + w_l^2 \le v_*^2 + w_*^2 \le v_l^2 + w_l^2, \tag{5.9}$$

followed by either a fast shock or a rarefaction connecting to (v_r, w_r).

In the latter case, we can impose across the nonclassical shock a kinetic relation of the form

$$-s\,\frac{1}{2}\big[v^2 + w^2\big] + \frac{3}{4}\big[\big(v^2 + w^2\big)^2\big] = \varphi(s), \tag{5.10}$$

where the kinetic function $\varphi(s)$ *satisfies the property* $(s \ge 0)$

$$\begin{aligned} -\frac{3}{4}s^2 \;&\le\; \varphi(s) \;\le\; 0, \\ \frac{d\varphi}{ds}(s) \;&\le\; 0. \end{aligned} \tag{5.11}$$

A unique solution is selected by (5.10) *in the one-parameter family of solutions. This solution depends continuously (in the* L^1 *norm) on its end states for* coplanar *initial data.*

We refer to Hayes and LeFloch [22] for further details on the Riemann solution to the cubic conservation law (5.3). A solution to (5.1) using only classical shock waves always exists. We emphasize that the one-parameter family of solutions constructed in Theorem 5.1 includes as special cases the classical Riemann solution (defined from the Oleinik criterion) and the Riemann solution using a rotational discontinuity followed by a fast shock. For noncoplanar data, the Riemann solution constructed in Theorem 5.1 does not depend continuously upon its initial states. (Consider "quasi-coplanar" initial data.) It is conceivable that this lack of continuity may be related to physical instabilities in MHD fluid which cannot be fully described by the model (5.1).

The coplanar discontinuities connecting (r_L, θ_L) to (r, θ) with $r \in (0, r_L/2)$ and $\theta = \theta_L + \pi$ are *overcompressive* shock waves. They possess nonunique traveling wave profiles, due to the existence of a component θ. When viewed as shock to the underlying scalar cubic conservation law, they are *classical* shocks, however.

5.3. Convergence result. As we now demonstrate it numerically in section 5.4 below, the solutions found in Theorem 5.1 may arise from diffusive-dispersive regularizations of (5.1). We consider here the system ($\epsilon > 0$, $\alpha \in \mathbb{R}$)

$$\begin{aligned} \partial_t v + \partial_x(v\,(v^2 + w^2)) &= \epsilon\,\partial_{xx} v + \alpha\,\epsilon\,\partial_{xx} w, \\ \partial_t w + \partial_x(w\,(v^2 + w^2)) &= \epsilon\,\partial_{xx} w - \alpha\,\epsilon\,\partial_{xx} v, \end{aligned} \tag{5.12}$$

called the derivative nonlinear Schrödinger–Burgers system. The right-hand side of (5.12) represents diffusive-dispersive effects arising in magnetic fluids due to the so-called Hall effect. When the ion inertia dispersion α can be neglected, (5.12) reduces to the Cohen–Kulsrud–Burgers (CKB) equations and converges, as $\epsilon \to 0$, to classical solutions. When $\alpha \neq 0$, the operator $\alpha\,\partial_{xx}$ in the right-hand side of (5.12) generates dispersion effect and nonclassical solutions may be obtained.

THEOREM 5.2. (1) *Let* $\big(v^\epsilon, w^\epsilon\big)$ *with* $\alpha \in (-1,1)$ *fixed be a family of solutions to* (5.12) *assuming at* $t=0$ *a Cauchy data* $\big(v_0^\epsilon, w_0^\epsilon\big)$ *such that*

$$v_0^\epsilon, w_0^\epsilon \in L^2(\mathbb{R}) \cap L^4(\mathbb{R}) \tag{5.13}$$

uniformly in ϵ. *Then* $\big(v^\epsilon, w^\epsilon\big)$ *is bounded in* $L^\infty\big(\mathbb{R}_+, L^2(\mathbb{R}) \cap L^4(\mathbb{R})\big)$ *and converges almost everywhere to a limiting function* (v,w), *a solution to* (5.1) *in the sense of distributions.*

(2) *The pair* $(U,F) = \big((v^2+w^2)/2,\, 3\,(v^2+w^2)^2/4\big)$ *is compatible in the sense* (2.3) *with the diffusive-dispersive regularization* (5.12). *Limits of traveling wave solutions to* (5.12) *additionally satisfy the entropy inequality* (5.8).

Proof of Theorem 5.2. The proof relies on the compensated compactness method of DiPerna [13] and more specifically the results in Chen [6]. We restrict attention to deriving the main a priori estimates needed in applying the theory, referring to [6, 13] for the details. The following entropy balance follows by multiplying the equations in (5.12) by v and w, respectively:

$$\begin{aligned}\tfrac{1}{2}\partial_t\big(v^2+w^2\big) + \tfrac{3}{4}\partial_x\big(v^2+w^2\big)^2 \;=\; & -\epsilon\,|\partial_x v|^2 - \epsilon\,|\partial_x w|^2 \\ & +\epsilon\,\partial_x\big(v\,\partial_x v + w\,\partial_x w\big) + \alpha\,\epsilon\,\partial_x\big(v\,\partial_x w - w\,\partial_x v\big).\end{aligned} \tag{5.14}$$

Integrating (5.14) over $(0,T)\times\mathbb{R}$ yields

$$\int_{\mathbb{R}} \frac{1}{2}\,\big(v^2+w^2\big)(T)\,dx + \int_0^T\!\!\int_{\mathbb{R}} \epsilon\,\big(|\partial_x v|^2 + |\partial_x w|^2\big)\,dxdt \;\le\; \int_{\mathbb{R}} \frac{1}{2}\,\big(v^2+w^2\big)(0)\,dx \;\le\; O(1). \tag{5.15}$$

Observe that the dispersive terms canceled out in this derivation, so that the estimate (5.15) does not depend on the coefficient $\alpha \in \mathbb{R}$.

We now multiply (5.14) on both sides by v^2+w^2 and integrate over $\mathbb{R}$ to get

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}} \frac{1}{4}\big(v^2+w^2\big)^2\,dx + \int_{\mathbb{R}} \frac{1}{2}\partial_x\big(v^2+w^2\big)^3\,dx \\ &\quad= -\int_{\mathbb{R}} \epsilon\,\big(v^2+w^2\big)\,\big(|\partial_x v|^2 + |\partial_x w|^2\big)\,dx - \int_{\mathbb{R}} \epsilon\,\big|\partial_x\big(v^2+w^2\big)\big|^2\,dx \\ &\qquad+ \int_{\mathbb{R}} \alpha\,\epsilon\,\big(v^2+w^2\big)\,\partial_x\big(v\,\partial_x w - w\,\partial_x v\big)\,dx. \end{aligned} \tag{5.16}$$

Thus we obtain

$$\begin{aligned} &\int_{\mathbb{R}} \frac{1}{4}\big(v^2+w^2\big)^2(T)\,dx + \int_0^T\!\!\int_{\mathbb{R}} \epsilon\,\big(v^2+w^2\big)\,\big(|\partial_x v|^2 + |\partial_x w|^2\big)\,dxdt \\ &+ \int_0^T\!\!\int_{\mathbb{R}} \epsilon\,\big|\partial_x\big(v^2+w^2\big)\big|^2\,dxdt \\ &= \int_{\mathbb{R}} \frac{1}{4}\big(v^2+w^2\big)^2(0)\,dx + \int_0^T\!\!\int_{\mathbb{R}} \alpha\,\epsilon\,\big(v^2+w^2\big)\,\big(v\,\partial_x w - w\,\partial_x v\big)\,dxdt. \end{aligned} \tag{5.17}$$

When $|\alpha| < 1$, the integrand of the last term in the right-hand side of (5.17) can be estimated by integrands of the left-hand side, namely,

$$\begin{aligned}\big|\alpha\,\epsilon\,\partial_x\big(v^2+w^2\big)\,\big(v\,\partial_x w - w\,\partial_x v\big)\big| &\le \frac{\alpha}{2}\,\epsilon\,\big|\partial_x\big(v^2+w^2\big)\big|^2 + \frac{\alpha}{2}\,\epsilon\,\big|v\,\partial_x w - w\,\partial_x v\big|^2 \\ &\le \frac{\alpha}{2}\,\epsilon\,\big|\partial_x\big(v^2+w^2\big)\big|^2 + \alpha\,\epsilon\,\big|v\,\partial_x w\big|^2 + \alpha\,\epsilon\,\big|w\,\partial_x v\big|^2 \\ &\le \frac{\alpha}{2}\,\epsilon\,\big|\partial_x\big(v^2+w^2\big)\big|^2 \\ &\quad + \alpha\,\epsilon\,\big(v^2+w^2\big)\,\big(|\partial_x v|^2 + |\partial_x w|^2\big).\end{aligned}$$

Therefore (5.17) implies

$$\begin{aligned}\int_{\mathbb{R}} \frac{1}{4}\big(v^2+w^2\big)^2(T)\,dx + \int_0^T\!\!\int_{\mathbb{R}} \big(1-\alpha/2\big)\,\epsilon\,\big(v^2+w^2\big)\,\big(|\partial_x v|^2 + |\partial_x w|^2\big)\,dxdt \\ + \int_0^T\!\!\int_{\mathbb{R}} \big(1-\alpha\big)\,\epsilon\,\big|\partial_x\big(v^2+w^2\big)\big|^2\,dxdt \;\le\; 0.\end{aligned} \tag{5.18}$$

The estimates (5.15) and (5.18) provide L^p uniform bounds for v^ϵ and w^ϵ, together with some derivative estimates. These estimates can be used along the lines of the proof in Schonbek [53] (and [22]) to show that a Young measure associated with (v^ϵ, w^ϵ) satisfies Tartar's commutation equation. The reduction theorem in [6] may be extended to L^p Young measures and shows that

$$\begin{aligned} &v^\epsilon \to v, \qquad w^\epsilon \to w \qquad &&\text{in the weak sense,} \\ &v_\epsilon^2 + w_\epsilon^2 \to v^2 + w^2 &&\text{in the strong sense.}\end{aligned} \tag{5.19}$$

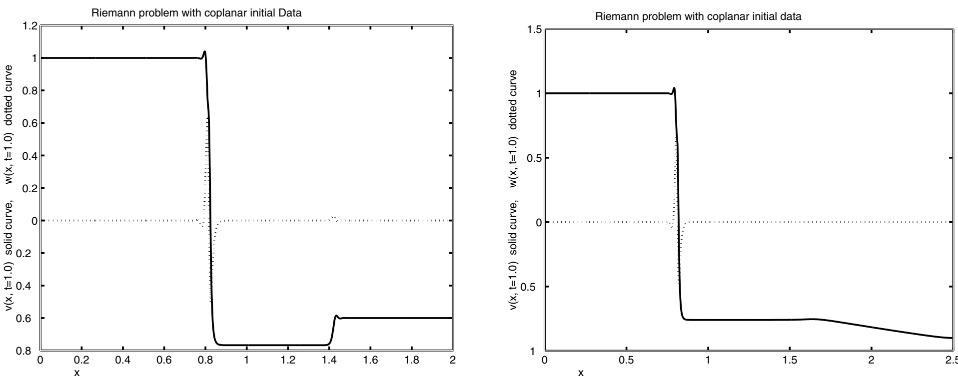
One can pass to the limit in (5.12) and deduce (5.1) as $\epsilon \to 0$.

Item (2) of the theorem follows from (5.14) and the uniform estimates (5.18). Observe that the first two terms in the right-hand side of (5.14) are nonpositive and converge to nonnegative bounded measures. The third term converges to zero in the sense of distributions. On the other hand the last term in the right-hand side of (5.14), due to the dispersive terms in (5.12), does not contribute to the dissipation measure (for the quadratic entropy only); namely, for each smooth function θ with compact support, one has

$$\begin{aligned}\left|\int_0^T\!\!\int_{\mathbb{R}} \epsilon\,\partial_x\big(v\,\partial_x w - w\,\partial_x v\big)\,\theta\,dxdt\right| &\le \int_0^T\!\!\int_{\mathbb{R}} \epsilon\,\big|v\,\partial_x w\big|\,\partial_x\theta\,dxdt \\ &\quad + \int_0^T\!\!\int_{\mathbb{R}} \epsilon\,\big|w\,\partial_x v\big|\,\partial_x\theta\,dxdt \\ &\le O(1)\,\epsilon\,\|v\,\partial_x w\|_{L^2\left((0,T)\times\mathbb{R}\right)} \\ &\quad + O(1)\,\epsilon\,\|v\,\partial_x w\|_{L^2\left((0,T)\times\mathbb{R}\right)} \\ &\le O(1)\,\epsilon^{1/2} \;\to\; 0.\end{aligned}$$

This completes the proof of Theorem 5.2. □

5.4. Numerical experiments. For coplanar initial data, we numerically demonstrate the existence of nonclassical shocks. We employ the following semidiscrete



(a) (b)

FIG. 5.1. *The slow shock is nonclassical: it cannot be a rotational wave, since across this shock,* $|u_m|^2 < |u_l|^2$.

approximation to the system (5.12):

$$
\begin{aligned}
&\frac{dv_k}{dt} + \frac{1}{2\,\Delta}\big(v_{k+1}\,(v_{k+1}^2 + w_{k+1}^2) - v_{k-1}\,(v_{k-1}^2 + w_{k-1}^2)\big)\\
&\qquad\qquad = \frac{\epsilon}{\Delta^2}\big(v_{k+1} - 2\,v_k + v_{k-1}\big) + \frac{\alpha\,\epsilon}{\Delta^2}\big(w_{k+1} - 2\,w_k + w_{k-1}\big),\\
&\frac{dw_k}{dt} + \frac{1}{2\,\Delta}\big(w_{k+1}\,(v_{k+1}^2 + w_{k+1}^2) - w_{k-1}\,(v_{k-1}^2 + w_{k-1}^2)\big)\\
&\qquad\qquad = \frac{\epsilon}{\Delta^2}\big(w_{k+1} - 2\,w_k + w_{k-1}\big) - \frac{\alpha\,\epsilon}{\Delta^2}\big(v_{k+1} - 2\,v_k + v_{k-1}\big)
\end{aligned}
\tag{5.20}
$$

for functions $v_k(t)$ and $w_k(t)$, where Δ denotes the spatial mesh-size. We integrate this system of ODEs in the same fashion as in subsection 4.2. The Riemann initial data for the numerical scheme are implemented as $(v_k(0), w_k(0)) = (v_l, w_l)$ for $k \le 0$ and (v_r, w_r) for $k > 0$.

In Figure 5.1, we plot the numerical results for two different coplanar data. The parameters are chosen to be $\Delta = 1/400$, $\epsilon = 1/800$, and $\alpha = 5/2$. In Figure 5.1(a), we use the initial data $(v_l, w_l) = (1, 0)$ and $(v_r, w_r) = (-.6, 0)$. The solid and the dotted lines represent the v- and w-components of the solution at the time $t = 1$, respectively. In Figure 5.1(b), we picture the results obtained with, instead, $(v_r, w_r) = (-.85, 0)$.

Appendix: Proof of Lemma 2.3. We follow Liu in [42] and treat the case (2.10a). The case (2.10b) is entirely similar. The statement on the wave speed follows easily from our assumption that $\nabla\lambda_j \cdot r_j$ changes sign only once along a shock curve. Let us show that the shock speed satisfies similar properties. By differentiating the Rankine–Hugoniot relation (2.13), we get

$$
-\frac{\partial}{\partial \mu_j}\bar\lambda_j(u_0, w_j)(w_j - u_0) + \big(Df(w_j) - \bar\lambda_j(u_0, w_j)\big)\frac{dw_j}{d\mu_j} \;=\; 0. \tag{A.1}
$$

Using the decompositions

$$
w_j - u_0 = \sum_{k=1}^{N} \alpha_k(u_0, w_j)\, r_k(w_j)
$$

and

$$\frac{dw_j}{d\mu_j} = \sum_{k=1}^{N} \beta_k(u_0, w_j)\, r_k(w_j),$$

we deduce that, for $k = 1, 2, \dots, N$,

$$-\frac{\partial}{\partial \mu_j}\bar\lambda_j(u_0, w_j)\,\alpha_k(u_0, w_j) + \big(\lambda_k(w_j) - \bar\lambda_j(u_0, w_j)\big)\,\beta_k(u_0, w_j) \;=\; 0.$$

In particular, for $k = j$,

$$\frac{\partial}{\partial \mu_j}\bar\lambda_j(u_0, w_j)\,\alpha_j(u_0, w_j) = \big(\lambda_j(w_j) - \bar\lambda_j(u_0, w_j)\big)\,\beta_j(u_0, w_j).$$

In view of our assumption (2.21), the coefficient $\alpha_j(u_0, w_j) = l_j(w_j)\cdot(w_j - u_0)$ has the same sign as $\mu_j - \mu_j(u_0)$, while $\beta_j(u_0, w_j) = l_j(w_j)\cdot dw_j/d\mu_j$ is strictly positive. Therefore for $\mu_j > \mu_j(u_0)$ we have

$$\tag{A.2} \begin{aligned} \frac{\partial}{\partial \mu_j}\bar\lambda_j(u_0, w_j) = 0 \text{ (resp., } > 0 \text{ or } < 0\big) \quad \text{iff} \\ \lambda_j(w_j) - \bar\lambda_j(u_0, w_j) = 0 \text{ (resp., } < 0 \text{ or } > 0\big), \end{aligned}$$

while for $\mu_j < \mu_j(u_0)$ the reversed inequalities are satisfied. Moreover it follows from (A.1) that (up to a multiplicative factor)

$$\tag{A.3} \frac{dw_j}{d\mu_j} \;=\; r_j(w_j) \quad \text{if} \quad \frac{\partial}{\partial \mu_j}\bar\lambda_j(u_0, w_j) \;=\; 0.$$

Denote by $\mu_j^\star(u_0)$ a point achieving the equality in (A.2). We now prove that, at the critical point $\mu_j = \mu_j^\star(u_0)$,

$$\tag{A.4} \begin{aligned} \frac{\partial^2}{\partial \mu_j^2}\bar\lambda_j(u_0, w_j) = 0 \text{ (resp., } > 0 \text{ or } < 0\big) \quad \text{iff} \\ \nabla\lambda_j(w_j)\cdot r_j(w_j) = 0 \,\text{(resp., } < 0 \text{ or } > 0\big) \end{aligned}$$

if $\mu_j^\star(u_0) > \mu_j(u_0)$, while the reversed inequalities are satisfied if $\mu_j^\star(u_0) < \mu_j(u_0)$. Namely, first rewrite the relation (A.1) (by using (A.3)) in the form

$$\tag{A.5} \begin{aligned} &\big(Df(w_j) - \bar\lambda_j(u_0, w_j)\big)\left(\frac{dw_j}{d\mu_j} - r_j(w_j)\right) \\ &\quad = \frac{\partial}{\partial \mu_j}\bar\lambda_j(u_0, w_j)(w_j - u_0) - \big(\lambda_j(w_j) - \bar\lambda_j(u_0, w_j)\big)\, r_j(w_j). \end{aligned}$$

Differentiating (A.5) once more, we obtain

$$\begin{aligned} &\frac{\partial}{\partial \mu_j}\big(Df(w_j) - \bar\lambda_j(u_0, w_j)\big)\left(\frac{dw_j}{d\mu_j} - r_j(w_j)\right) \\ &+ \big(Df(w_j) - \bar\lambda_j(u_0, w_j)\big)\left(\frac{d^2 w_j}{d\mu_j^2} - \frac{\partial}{\partial \mu_j} r_j(w_j)\right) \\ &= \frac{\partial^2}{\partial \mu_j^2}\bar\lambda_j(u_0, w_j)(w_j - u_0) + \frac{\partial}{\partial \mu_j}\bar\lambda_j(u_0, w_j)\frac{dw_j}{d\mu_j} \\ &- \frac{\partial}{\partial \mu_j}\big(\lambda_j(w_j) - \bar\lambda_j(u_0, w_j)\big)\, r_j(w_j) - \big(\lambda_j(w_j) - \bar\lambda_j(u_0, w_j)\big)\,\frac{\partial}{\partial \mu_j} r_j(w_j). \end{aligned}$$

Plugging the value $\mu_j = \mu_j^\star(u_0)$ in the above formula and using (A.2)–(A.3), we obtain

$$\big(Df(w_j) - \bar\lambda_j(u_0, w_j)\big)\left(\frac{d^2 w_j}{d\mu_j^2} - \frac{\partial}{\partial \mu_j} r_j(w_j)\right) = \frac{\partial^2}{\partial \mu_j^2}\bar\lambda_j(u_0, w_j)(w_j - u_0) - \frac{\partial}{\partial \mu_j}\lambda_j(w_j)\, r_j(w_j).$$

Multiplying the latter by $l_j(w_j)$ and observing that $\bar\lambda_j(u_0, w_j) = \lambda_j(w_j)$ so that the left-hand side vanishes, we arrive at

$$\begin{aligned}\frac{\partial^2}{\partial \mu_j^2}\bar\lambda_j(u_0, w_j)\, l_j(w_j)\cdot(w_j - u_0) &= \frac{\partial}{\partial \mu_j}\lambda_j(w_j)\\ &= \nabla\lambda_j(w_j)\cdot r_j(w_j).\end{aligned}$$

The desired result (A.4) follows immediately from the above formula and assumption (2.21ii).

We now use the notation $g(\mu_j) := \lambda_j(w_j(\mu_j; u_0))$ and $h(\mu_j) := \bar\lambda_j(u_0, w_j(\mu_j; u_0))$. The property (A.2) shows that (2.24a) is satisfied for values μ_j close enough to $\mu_j(u_0)$, at least. Consider the largest value $\mu_j < \mu_j(u_0)$ such that $h(\zeta_j) - g(\zeta_j) > 0$ holds for all $\zeta \in (\mu_j, \mu_j(u_0)$. Call this value $\mu_j^\star(u_0)$ and observe that $h(\mu_j^\star(u_0)) = g(\mu_j^\star(u_0))$. In view of (A.2) one also has $h'(\mu_j^\star(u_0)) = 0$.

Assume that $\mu_j^\star(u_0) > 0$. In view of (A.4), one has $h''(\mu_j^\star(u_0)) > 0$ since $\mu_j^\star(u_0) > 0$. Thus the function should decrease for $\mu_j < \mu_j^\star(u_0)$ at least in a small neighborhood of $\mu_j^\star(u_0)$. According to (A.2), the wave speed should then be above the shock speed in this range, and so the wave speed g should be nonincreasing. The function g is increasing near $\mu_j(u_0)$ and nonincreasing near $\mu_j^\star(u_0)$, so g must have a critical point in the interval $\big[\mu_j^\star(u_0), \mu_j(u_0)\big)$. Since the only critical point of the wave speed is $\mu_j = 0$ and $\mu_j^\star(u_0) > 0$ by assumption, we reach a contradiction. Henceforth, one must have $\mu_j^\star(u_0) \le 0$.

Finally the shock speed is monotone in the whole region $\mu_j < \mu_j^\star(u_0)$, since otherwise that would imply the existence of a critical point for the function g, which is not possible. This completes the proof of Lemma 2.3. ∎

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The authors learned that D. Marchesin, B. Plohr, and K. Zumbrun recently observed that undercompressive shocks may arise in strictly hyperbolic systems. In a recent and interesting preprint by H. Freistühler, the selection of undercompressive shocks in hyperbolic systems and their stability in the multidimensional setting is also investigated.

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