Permutation Patterns, Statistics On Permutations And Sorting By Shuffling

by

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THESIS

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To my grandparents Petrana and Stoyan,

for all the sacrifices they have made for me!

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SUMMARY

Patterns in permutations have been extensively studied in the past, as a topic within the area of enumerative combinatorics, and they have turned out to be useful when answering various questions in computer science, statistics, computational biology and other fields.

Suppose that we have two permutations $\sigma = \sigma_1 \cdots \sigma_k \in S_k$ and $\pi = \pi_1 \cdots \pi_n \in S_n$, where $k \leq n$ and S_m denotes the set of permutations of size m. We say that π *contains* the *classical pattern* σ, if there exist indices $1 \leq i_1 < \cdots < i_k \leq n$, such that $\pi_{i_a} < \pi_{i_b}$, if and only if $\sigma_a < \sigma_b$, for every $1 \leq a, b \leq k$. This means that the elements of the subsequence $\pi_{i_1} \cdots \pi_{i_k}$ are in the same relative order as the elements of σ. Every such subsequence of π is called an *occurrence* of σ in π. If π does not have any occurrences of σ, we say that π *avoids* σ. Two generalizations of classical patterns are the so-called *vincular* and *bivincular* patterns.

In this thesis, we consider a series of new questions whose answers involve permutation patterns. We investigate sorting by queues that can rearrange their content by applying permutations corresponding to shuffling methods. Some of our main results are related to sorting by cuts. We also study a large class of permutation statistics, which can be written as a linear combination of bivincular patterns. We develop an approach, previously applied to statistics in other combinatorial structures, which allows one to find closed-form expressions for the higher moments of the statistics in this class. In addition, we consider a generalization of vincular patterns, which we call *distant patterns*, and we obtain a number of interesting enumerative results related to them.

CHAPTER 1

INTRODUCTION AND RELEVANT RESULTS

If we have two combinatorial objects, it is natural to ask how many times does the first object, called a *pattern*, occur as a part of the second one. Patterns in various combinatorial structures have been extensively studied in the past. This includes patterns in set partitions [110], trees [45], Dyck paths [19] and permutations [26, 102]. In this thesis, we consider some new results involving the notion of patterns in permutations. We investigate a set of new questions related to sorting, as well as an application of this notion to statistics in permutations. Furthermore, we study a certain generalization of permutation patterns called *distant* patterns. In the current chapter, we recall some necessary background and some standard results related to permutation patterns.

1.1 Avoidance of classical patterns in permutations

The set of consecutive integers $\{i, i+1, \ldots, j\}$ will be denoted by $[i, j]$. A permutation of size n is a bijective map from $[n] := [1, n]$ to itself. When referring to permutations, we will use their one-line representation. The set of all permutations of size n will be denoted by S_n. For example, $S_1 = \{1\}$, $S_2 = \{12, 21\}$ and $S_3 = \{123, 132, 213, 231, 312, 321\}$.

Definition 1.1 (reduction of a sequence)**.** Let $q = q_1 \cdots q_k$ be a sequence of k different numbers. The *reduction* of q, denoted by $\mathrm{red}(q)$, is the unique permutation $\pi = \pi_1 \cdots \pi_k \in S_k$, such that its elements are in the same relative order as the elements of q, i.e., $\pi_i < \pi_j$ if and only if $q_i < q_j$, for all $i, j \in [k]$.

The permutation $\text{red}(q)$ can be obtained by replacing the i-th smallest element of q with i, for every $i \in [k]$. For example, $\text{red}(5724) = 3412$ since the smallest element, 2, of 5724 is replaced with 1, the second smallest element, 4, is replaced by 2, etc.

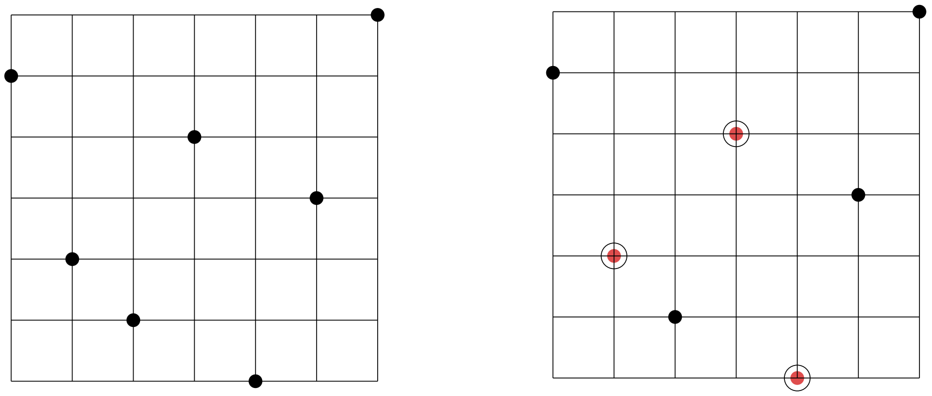
We say that α is a subsequence of the permutation $\pi = \pi_1 \cdots \pi_n$, if there exist indices $1 \le i_1 < \cdots < i_m \le n$, such that $\alpha = \pi_{i_1}\pi_{i_2}\cdots\pi_{i_m}$. If the indices $i_1, \ldots, i_m$ are consecutive numbers, then we will say that α is a *segment* of π.

Definition 1.2 (occurrence of a classical pattern)**.** A permutation π contains a permutation σ as a (classical) pattern, if there is a subsequence λ of π such that $\text{red}(\lambda) = \sigma$. We will say that λ is an *occurrence* of σ in π. If π does not contain σ, then π *avoids* σ.

In other words, π contains the pattern σ, if and only if π has a subsequence with elements in the same relative order as the elements of σ.

Example 1.3. The permutation $\pi = 6325147$ contains the pattern $\sigma = 231$ since 351 is a subsequence of π and $\text{red}(351) = 231$. It avoids the pattern 1234, as π does not have an increasing subsequence of size 4.

The given definition of pattern occurrence is related to the one-line representation of a permutation. An alternative geometric definition provides another perspective. We can visualize the permutation $\pi = \pi_1\pi_2\cdots\pi_n \in S_n$ by plotting the points (i, π_i) in the xy-plane. This is how we obtain the *permutation diagram* of π. A permutation π contains a permutation σ as a pattern, if the diagram of π contains the diagram of σ as a subdiagram. The permutation diagram of 6325147 is shown in Figure 1a. This diagram contains the diagram of 231, as shown in Figure 1b. Therefore, 6325147 contains the pattern 231.



(a) The permutation diagram of 6325147.

(b) 6325147 contains the pattern 231.

Figure 1: Occurrence of a pattern.

Finding the set of permutations (or the number of permutations) of a certain size with a given number of occurrences of a fixed pattern is a difficult problem, in general. Thus, most of the existing literature focuses on questions related to avoidance/containment of permutation patterns. If X is a set of patterns, then let

$$Av_n(X) := \{\pi \mid \pi \in S_n,\ \pi \text{ avoids } \sigma \text{ for any } \sigma \in X\},$$

with $av_n(X) := |Av_n(X)|$. Also, let $Av(X) := \cup_{n=1}^{\infty} Av_n(X)$, for any set of permutations X. We will also write

$$C_n(X) := S_n \setminus Av_n(X),$$

to denote the set of permutations of size n that contain at least one of the patterns in X.

Definition 1.4 (permutation class)**.** A *permutation class* C is a set of permutations, such that if $\pi \in C$ and π contains σ, then $\sigma \in C$.

Using that definition, one can check directly that the set $Av_n(X)$ is a permutation class, for any set X. It is not difficult to prove that all permutation classes are of this kind.

Theorem 1.5. For every permutation class C, there exists a unique set of patterns T, such that

$$C = Av(T).$$

The set T is called a *basis* of C.

Proof. Let $\overline{C} := \cup_{n=1}^{\infty} S_n \setminus C$. Consider the set $T := \{t \in \overline{C} \mid \text{for all } \sigma, \text{if } t \text{ contains } \sigma \text{ then } \sigma \notin \overline{C}\}$. In other words, T is the set of those permutations that are not in C, and which are the minimal elements in terms of pattern containment. One can verify directly that $C = Av(T)$. □

There are well-known results in combinatorics that can be formulated in terms of permutation patterns. For example, it is easy to see that the following theorem of Erdős and Szekeres states that $av_n(12\cdots r, s\cdots 1) = 0$, for any $r, s \geq 1$, such that $n \geq rs + 1$.

Theorem 1.6 (Erdős and Szekeres [59]). For $r, s \in \mathbb{N}$, any sequence of distinct real numbers of size at least $rs + 1$ contains an increasing subsequence of size r or a decreasing subsequence of size s.

Some of the first results related to permutation patterns can be traced back to work by Euler in the 18th century [62] and work by Percy MacMahon at the beginning of the 20th century [107]. MacMahon obtained the generating function for the distribution of the number of inversions in permutations, which corresponds to the number of occurrences of the classical pattern 21. In 1970s, results by Knuth [104] showed that permutation patterns can appear in facts related to computer science. The article of Rodica Simion and Frank Schmidt from 1985 [129] was the first one to consider the simultaneous avoidance of multiple patterns and the first systematic study of permutation patterns. More details on the history of the first results in permutation patterns and their applications can be found in the Forward of [102].

1.1.1 Symmetry classes and Wilf-equivalent classes of permutations

For a permutation $\pi = \pi_1 \cdots \pi_n$, let π^{-1} denotes the *inverse* of π. Also, let $\overline{\pi}$ and π^r denote the *complement* and the *reverse* of π, respectively:

$$\overline{\pi}_j = n + 1 - \pi_j,$$

and

$$(\pi^r)_j = \pi_{n+1-j},$$

for every $j \in [n]$.

These three transformations are simple bijective maps on S_n, which generate the *dihedral group* D_4; i.e., the group of symmetries of the square, if we look at the corresponding transformations of the permutation diagram of π. In particular, the reverse, the complement and the inverse map act as a horizontal, vertical and diagonal symmetry of the square, respectively (see Figure 2 below).

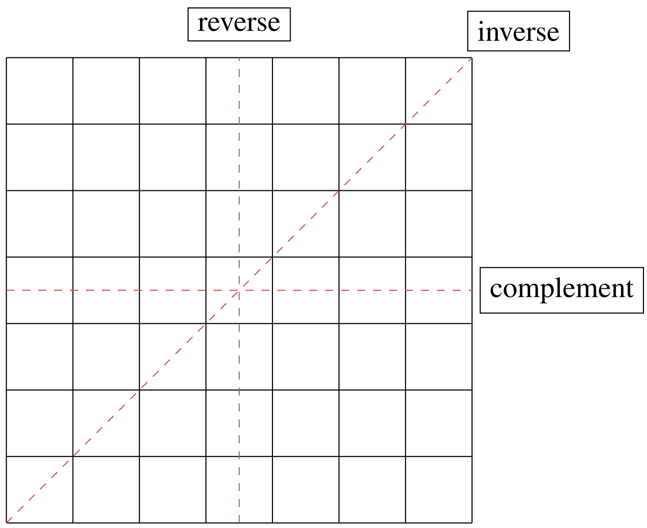


Figure 2: The symmetries corresponding to the three transformations generating the group D_4.

Formally, if D_4 is the group of symmetries of the square and if i, r and c denote the inverse, the reverse and the complement maps, respectively, then

$$D_4 = < i, r, c > = \{id, i, r, c, i \circ r, c \circ r, i \circ c, i \circ c \circ r\}.$$

The action of D_4 partitions S_n into several equivalence classes that will be called *symmetry classes*.

Definition 1.7 (symmetry class of a permutation)**.** The symmetry class of a permutation σ is

$$[\sigma] := \{f(\sigma) \mid f \in D_4\}.$$

For example, $[132] = \{132, 231, 213, 312\}$. Table I below lists the symmetry classes partitioning S_4.

Symmetry class	Permutations in the class
$[1234]$	$\{1234, 4321\}$
$[1243]$	$\{1243, 3421, 4312, 2134\}$
$[1324]$	$\{1324, 4231\}$
$[1342]$	$\{1342, 2431, 4213, 3124, 1423, 3241, 4132, 2314\}$
$[1432]$	$\{1432, 2341, 4123, 3214\}$
$[2143]$	$\{2143, 3412\}$
$[2413]$	$\{2413, 3142\}$

TABLE I: The 7 symmetry classes for the permutations of size 4.

Definition 1.8 (the symmetry class of a set)**.** The symmetry class of a set of permutations X, denoted by $[X]$, is the collection of sets obtained from X through the action of D_4:

$$[X] := \{\{f(\pi) \mid \pi \in X\} \mid f \in D_4\}.$$

Example 1.9.

$$[132, 4321] = \{\{132, 4321\}, \{231, 1234\}, \{213, 4321\}, \{312, 1234\}\},$$

since $id(\{132, 4321\}) = i(\{132, 4321\}) = \{132, 4321\}$, $r(\{132, 4321\}) = i(c(\{132, 4321\})) = \{231, 1234\}$, $c(\{132, 4321\}) = i(r(\{132, 4321\})) = \{312, 1234\}$, and $c(r(\{132, 4321\})) = i(c(r(\{132, 4321\}))) = \{213, 4321\}$.

Theorem 1.10. For every set of patterns X and any $f \in D_4$,

$$Av(f(X)) = f(Av(X)).$$

Proof. Note that $\sigma \in Av(\pi)$ if and only if $f(\sigma) \in Av(f(\pi))$. From here, we see that $\sigma \in Av(X)$ if and only if $f(\sigma) \in Av(f(X))$. Thus $f(Av(X)) = Av(f(X))$. □

Definition 1.11 (symmetry-equivalent sets)**.** Two sets of permutations, X_1 and X_2, are *symmetry-equivalent* if they belong to the same symmetry class, i.e., if there exists $f \in D_4$, such that $f(X_1) = X_2$. We will write $X_1 \cong X_2$.

For example, $\{132, 4321\} \cong \{231, 1234\}$, because $r(\{132, 4321\}) = \{231, 1234\}$.

Theorem 1.12. If X_1 and X_2 are two sets of permutations, then $X_1 \cong X_2$ if and only if $Av(X_1) \cong Av(X_2)$.

Proof. $X_1 \cong X_2$ if and only if $f(X_1) = X_2$, for some $f \in D_4$. Therefore, $Av(f(X_1)) = Av(X_2)$. By Theorem 1.10, $Av(f(X_1)) = f(Av(X_1)) = Av(X_1)$. □

Definition 1.13 (Wilf-equivalent classes)**.** Two permutation classes $Av(X_1)$ and $Av(X_2)$ are *Wilf-equivalent* if for every $n \geq 1$,

$$av_n(X_1) = av_n(X_2).$$

We will write $Av(X_1) \backsim Av(X_2)$.

Theorem 1.14. If $X_1 \cong X_2$, then $Av(X_1) \backsim Av(X_2)$.

Proof. We know that $Av(X_1) \cong Av(X_2)$ and thus $f(Av(X_1)) = Av(X_2)$ for some $f \in D_4$. Therefore, $f(Av_n(X_1)) = Av_n(X_2)$, for every $n \geq 1$, and $|Av_n(X_1)| = |Av_n(X_2)|$, since f is a bijection. $\square$

Note that the converse of Theorem 1.14 does not hold, i.e., $Av(X_1) \backsim Av(X_2)$ does not imply that X_1 and X_2 are symmetry-equivalent. For instance, the classes $Av(\{1234\})$ and $Av(\{1432\})$ are Wilf-equivalent [131], but they are not symmetry-equivalent.

1.1.2 Enumerative results

Most of the permutation pattern research is about enumerating the sets $Av_n(X)$, that is, finding a closed-form formula, generating function or a recurrence for $av_n(X)$, for various sets X. Here, the symmetries discussed in the previous section often allow us to reduce this enumeration problem to the enumeration of other permutation classes. Below, we consider the cases when $|X| = 1$. Let $X = \{q\}$, where q is a classical permutation pattern, with $|q| = k$. Instead of $Av_n(\{q\})$ and $av_n(\{q\})$, we will write $Av_n(q)$ and $av_n(q)$.

1. If $k = 1$, then $q = 1$ and obviously $av_n(1) = 0$, for $n \geq 1$, since every permutation of positive size has at least one element.

2. If $k = 2$, then $q = 12$ or $q = 21$. We have

$$av_n(12) = av_n(21) = 1,$$

for every $n \geq 1$, since every permutation of size n except $12 \cdots n$ contains an occurrence of 21, i.e., an inversion. Furthermore, 21 is the reverse of 12, and thus we have only one Wilf-equivalent class.

3. If $k = 3$, then we have two symmetry classes: $\{123, 321\}$ and $\{132, 312, 213, 231\}$. The following fact can be derived from results of Percy MacMahon back in 1915 [108], as well as from later results of Schensted around 1960 [128] and Hammersley around 1970 [81].

Theorem 1.15.

$$av_n(123) = C_n,$$

where $C_n = \frac{\binom{2n}{n}}{n+1}$ is the n-th Catalan number.

In fact, there are the same number of 132-avoiding permutations of size n. Thus we have only one Wilf-equivalent class of patterns of size 3.

Theorem 1.16.

$$av_n(123) = av_n(132) = C_n.$$

The first bijection between permutations avoiding a pattern from $\{123, 321\}$ and permutations avoiding a pattern from $\{132, 312, 213, 231\}$ was given by Knuth [104]. Several other bijections of this kind, including one by Simion and Schmidt [129], were later found by various authors. [102, Chapter 4] contains detailed description for each of them and a classification by the popular statistics that each bijection preserves.

4. If $k = 4$, then the standard bijections give us 7 symmetry classes (see Table I). West [141] showed that 1234, 1243 and 2143 belong to the same Wilf-equivalent class, while Stankova showed that $1342 \backsim 2413$ [130] and that $1234 \backsim 3214$ [131]. The listed results imply that in fact we have 3 different Wilf-equivalent classes of patterns of size 4. The classes of 1234 and 1342 are enumerated by Gessel [79] and Bóna [22], respectively. No one has yet enumerated the class of 1324, despite the recent interest [20, 39, 111]. This information is summarized in Table II below. Instead of the formula for $av_n(1234)$ found by Gessel in [79], we give an alternative form of the same formula in the table, which was found by Gessel a few years later and shared with Bóna in private communication [26]. The same formula was established independently by Bousquet-Mélou [31].

Class	Formula for $av_n(q)$	Reference
$\{1234, 1243, 2143, 3214\}$	$\frac{1}{(n+1)^2(n+2)} \sum_{k=1}^{n} \binom{2k}{k}\binom{n+1}{k+1}\binom{n+2}{k+1}$	[79, Gessel]
$\{1342, 2413\}$	$(-1)^{n-1}\frac{(7n^2-3n-2)}{2} + 3\sum_{i=2}^{n}(-1)^{n-i}2^{i+1}\frac{(2i-4)!}{i!(i-2)!}\binom{n-i+2}{2}$	[22, Bóna]
$\{1324\}$	not yet found	–

TABLE II: The 3 Wilf-equivalent classes for patterns of size 4.

The work of Stankova and West [132] completed the classification of patterns of size 6 and 7. The same article lists the Wilf-equivalent classes for the patterns of size 5.

A great number of other works investigated the enumeration of the class $Av_n(X)$, when $|X| > 1$. For example, all of the classes when X consists of two patterns of size 3 have been enumerated in the seminal work of Simion and Schmidt [129]. The classification when X is a set of two size-4 patterns has been completed by Le [106]. However, three of these classes are still not enumerated. The enumeration and classification of all triples of 4-letter patterns has been completed in the work of Callan et al. [35]. The recent work of Albert et al. [2] discusses a new algorithmic framework, which allows to find the generating functions of all symmetry classes avoiding m patterns of size 4, where $4 \leq m \leq 24$.

1.1.2.1 Separable permutations

Definition 1.17 (Direct sum and skew-sum)**.** If σ and τ are two permutations of sizes k and l, respectively, then their *direct sum* $\sigma \oplus \tau$ and their *skew-sum* $\sigma \ominus \tau$ are defined as follows:

$$(\sigma \oplus \tau)(i) = \begin{cases} \sigma(i), & \text{if } i \leq k, \\ k + \tau(i-k), & \text{if } k+1 \leq i \leq k+l. \end{cases} \qquad (\sigma \ominus \tau)(i) = \begin{cases} l + \sigma(i), & \text{if } i \leq k, \\ \tau(i-k), & \text{if } k+1 \leq i \leq k+l. \end{cases}$$

Example 1.18. $3124 \oplus 132 = 3124576$ and $3124 \ominus 132 = 6457132$.

Definition 1.19 (separable permutations)**.** The *separable permutations* are those which can be built from the permutation 1 by repeatedly applying the $\oplus$ and the $\ominus$ operations.

Separable permutations were introduced by Bose et al. [30]. In the same work, the authors proved the following characterization.

Theorem 1.20. The set of *separable* permutations is $Av(3142, 2413)$.

This important class of permutations arose in the study of pop-stack sorting [12], which will be discussed in Section 1.3.2.3. This class was first enumerated by West.

Theorem 1.21 (West [141])**.** The number of separable permutations of size n, $av_n(3142, 2413)$, is the $(n-1)$-st Schröder number [118, A006318].

1.1.3 Asymptotic results and the Stanley-Wilf conjecture

In general, it is very difficult to find a closed-form expression for $av_n(X)$, when the class X contains patterns of size 5 or more. Because of that, we are interested in obtaining results on the asymptotic behaviour of these numbers, since these provide intuition for the exact sizes of the sets $Av_n(X)$.

By Theorem 1.15, every classical pattern $q \in S_3$ is avoided by a number of permutations given by the Catalan number $C_n = \frac{\binom{2n}{n}}{n+1}$. We have $\frac{C_n}{C_{n-1}} = \frac{4n-2}{n+1}$, so $\lim_{n\to\infty} \frac{C_n}{C_{n-1}} = 4$ and therefore

$$\lim_{n\to\infty} \sqrt[n]{av_n(q)} = 4, \text{ for any } q \in S_3.$$

This leads naturally to the question of whether these limits exists for patterns q of larger size, and what are their values.

Using a simple observation, inspired by the proof of the result of Erdős and Szekeres (Theorem 1.6), Bóna [25] showed that $av_n(12\cdots k) \leq (k-1)^{2n}$. Then, Regev [126] obtained results implying the following.

Theorem 1.22 (Regev [126])**.**

$$\lim_{n\to\infty} \sqrt[n]{av_n(12\cdots k)} = (k-1)^2.$$

Unaware of the work of Regev and the bound on $av_n(12\cdots k)$, Herbert Wilf asked, around 1980, whether $av_n(\sigma) \leq (k-1)^n$ for every $\sigma \in S_k$. Independently, Richard Stanley asked whether $\lim_{n\to\infty} \sqrt[n]{av_n(\sigma)} = (k-1)^2$, for all $\sigma \in S_k$.

This raised the conjecture below.

Conjecture 1.23. (The Stanley-Wilf upper bound conjecture) For every $\sigma \in S_k$, there exists a real constant c_σ, such that $av_n(\sigma) \leq c_\sigma^n$.

Wilf formulated the following similar conjecture.

Conjecture 1.24. (The Stanley-Wilf limit conjecture) For every $\sigma \in S_k$, there exists a real constant c_σ, such that $\lim_{n\to\infty} \sqrt[n]{av_n(\sigma)} = c_\sigma$.

The limit conjecture easily implies the upper bound conjecture. In 1999, Arratia [7] showed that the converse is also true, i.e., the upper bound conjecture implies the limit conjecture and thus the two conjectures are equivalent. An affirmative answer to these equivalent questions became known as the Stanley-Wilf conjecture (a name coined by Bóna [135]). Different researchers attempted to prove it, but they were able to do that only in special cases. The conjecture was finally established in 2004 by Marcus and Tardos [116] who in fact proved a conjecture of Füredi and Hajnal, which had been shown earlier to imply the Stanley–Wilf conjecture. We refer to the articles of Egge [58] and Stanley [135] for more on the historical details and the facts leading to the proof.

The limit $c_\sigma = \lim_{n\to\infty} \sqrt[n]{av_n(\sigma)}$ is called a *Stanley-Wilf limit* of σ. As the Stanley-Wilf conjecture holds, it is natural to ask what are the possible values of these limits, as well as their values for particular patterns. It is interesting to note that the largest and the smallest values of c_σ for a fixed $k = |\sigma|$ are not achieved by the identity permutation. For example, Theorem 1.22 implies that $c_{1234} = 9$. However, the work of Bóna [22] gives that $c_{1342} = 8$ and the work of Albert et al. [4] gave that $c_{1324} > 9.47$. Currently, the best known bound is $c_{1324} > 10.27$ by Bevan et al. [20]. Thus c_{1234} is neither the largest nor the smallest Stanley-Wilf limit for the patterns in S_4. Bóna [23] conjectured that among the patterns of a given size, the largest Stanley-Wilf limit is attained by a layered permutation. In the same work, he also showed that Stanley-Wilf limits are not necessarily rational numbers. Jacob Fox [75] refuted the conjecture of Bóna by also showing that $c_\sigma = 2^{k^{\Theta(1)}}$, for almost all permutations σ on k letters.

Finally, it is important to note that it is not known whether $\lim_{n\to\infty} \sqrt[n]{av_n(\Pi)}$ exists, for every set of patterns Π.

1.2 Vincular and consecutive patterns

In this section, we introduce other types of permutation patterns that are needed to describe several of our results in Chapter 3 and Chapter 4 of the thesis. First, we define *vincular patterns*, which allow one to require the numbers in every pattern occurrence, corresponding to letters at consecutive positions in the pattern, to be at consecutive positions in the permutation.

Definition 1.25 (vincular patterns)**.**

(i) A *vincular pattern* σ of size k is a permutation in S_k some of whose consecutive elements can be underlined.

(ii) An occurrence of the vincular pattern $\sigma \in S_k$ in the permutation π is a subsequence $\lambda = \lambda_1\lambda_2\cdots\lambda_k$ of π, such that λ is an occurrence of σ in π, as a classical pattern, and the numbers $\lambda_i, \lambda_{i+1}, \ldots, \lambda_j$ are at consecutive positions in π, for any underlined segment $\sigma_i\sigma_{i+1}\cdots\sigma_j$ in σ.

Example 1.26. $3\underline{12}$ is a vincular pattern of size 3. 615 is an occurrence of $3\underline{12}$ in the permutation 621543 since the numbers 1 and 5 are at consecutive positions in it and $\mathrm{red}(615) = 312$. Meanwhile, 614 is not an occurrence of $3\underline{12}$.

Vincular patterns were introduced by Babson and Steingrímsson under the name "generalized patterns" [13]. It was shown in the same article that many *Mahonian statistics* in permutations can be written as linear combinations of such patterns. A Mahonian statistic is a function defined over permutations, whose values have the same distribution over S_n, for each n, as the number of inversions. Several subsequent works investigated vincular patterns in their own right (see the survey of Steingrímsson [137]). These patterns were also called "dashed patterns" to distinguish them from other generalizations of classical patterns. According to [15], Claesson was the first one to use the term *vincular* patterns, to connect them with the bivincular patterns defined in Section 1.5.

Definition 1.27 (consecutive patterns)**.** A *consecutive* pattern is a vincular pattern with all of its elements underlined. That is, in an occurrence of a consecutive pattern, the pattern must appear as a consecutive substring in the permutation.

Example 1.28. 621543 contains a $\underline{321}$ pattern, but it does not contain a $\underline{312}$ pattern.

Some well-known counting problems in permutations can be restated as problems about counting occurrences of consecutive patterns. For instance, occurrences of $\underline{12}$ (and respectively $\underline{21}$) correspond to ascents (respectively, descents) and are counted by the Eulerian numbers [119]. Another example are the *up-and-down* permutations, which are those in $S_n(\underline{123}, \underline{321})$.

A popular and important reference on consecutive patterns is the work of Elizalde and Noy [57]. There, they give generating functions for the number of permutations with a given number of occurrences of several consecutive patterns of size 3 and 4, as well as of certain patterns of arbitrary size. Note that this is a more general problem than determining the number of permutations avoiding (having 0 occurrences of) a consecutive pattern.

1.3 Sorting devices

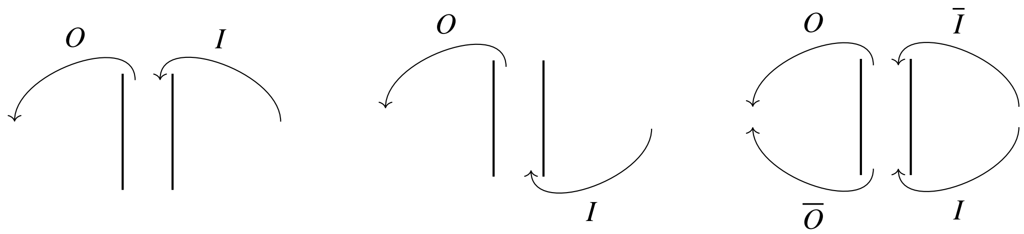
A sorting device $\mathbb{D}$ is a tool that transforms a given input permutation π by following a particular algorithm which could be deterministic or non-deterministic. The result is an output permutation π'. During the execution of the algorithm, every device $\mathbb{D}$ has a given configuration $(s_{inp}, s_{dev}, s_{out})$, comprised of three sequences (strings) corresponding to the current string in the input, in the device, and in the output, respectively. The initial configuration is $(\pi, \varepsilon, \varepsilon)$ and the final configuration is $(\varepsilon, \varepsilon, \pi')$.

Denote by $\mathbb{D}(\pi)$ the set of possible output permutations π', when using the device $\mathbb{D}$ on input π. If id_n denotes the identity permutation of size n, then we say that a permutation π can be *sorted* by $\mathbb{D}$ if $id_n \in \mathbb{D}(\pi)$, i.e., if there exists a sequence of input and output operations over the device $\mathbb{D}$, which transforms π to the identity. Let

$$S_n(\mathbb{D}) := \{\pi \mid \pi \in S_n, id_n \in \mathbb{D}(\pi)\}$$

be the set of the permutations sortable by $\mathbb{D}$. Furthermore, let $p_n(\mathbb{D}) := |S_n(\mathbb{D})|$. In general, if $\mathbb{D}$ is a sorting device and $\pi' \in \mathbb{D}(\pi)$, then any sequence of configurations for $\mathbb{D}$ that begins with $(\pi, \varepsilon, \varepsilon)$ and ends with $(\varepsilon, \varepsilon, \pi')$, together with the sequence of corresponding sorting operations, will be called an *iteration* of $\mathbb{D}$ over the input π.

A natural question when one has a sorting device is: “Which $\pi \in S_n$ can be sorted when we use this device?” Donald Knuth [104, Chapter 2.2.1] asked this question for the classical data structures *stack*, *queue* and *deque* (double-ended queue) shown in Figure 3.



(a) stack (b) queue (c) deque

Figure 3: The input and output operations on stack, queue and deque.

These three data structures are linear lists which are used frequently in programming to store and access data. For each device, we have input operations (also called *push* operations), which insert an element from the input to the device and output operations (also called *pop* operations), which move an element from the device to the output:

- stack ($\mathbb{S}\mathbb{t}$): the input operations I and the output operations O are made at one end of the list.
- queue ($\mathbb{Q}$): the input operations I are made at one end of the list and the output operations O are made at the other end of the list.

- deque ($\mathbb{Deq}$): two kinds of input operations (I and $\overline{I}$) exist, as well as two kinds of output operations (O and $\overline{O}$). The two pairs of input and output operations are made at the two opposite ends of the list, as shown in Figure 3c.

In fact, Knuth asked which permutations can be obtained using these three devices, if we begin with the identity permutation $12\cdots n$. The two questions correspond to two equivalent viewpoints, since a permutation π can be obtained from the identity by applying a given sequence of operations, if and only if π^{-1} is sorted by the same sequence of operations.

1.3.1 Permutations that can be sorted by stack, queue and deque

Figure 4 given below shows an example of a permutation that can be sorted by a stack. Each of the subfigures of Figure 4 shows the corresponding configuration of the stack, as well as the operation that is used. For instance, the configuration of the stack corresponding to Figure 4e is $(3, 42, 1)$ since the input contains the string "3", the content of the stack contains the string "42", and the output contains the string "1". Recall that the initial configuration shown in Figure 4a is $(4213, \varepsilon, \varepsilon)$, while the final configuration shown in Figure 4i is $(\varepsilon, \varepsilon, 1234)$. The set of permutations that can be sorted with a stack turns out to be a permutation class.

Theorem 1.29. (Knuth [104, Chapter 2.2.1, Exercise 2])

$$S_n(\mathbb{St}) = Av(231).$$

We see that we cannot sort all the permutations with a stack. This is not possible even with a deque. Pratt [123] showed that the set of the deque sortable permutations is a permutation class, which has an

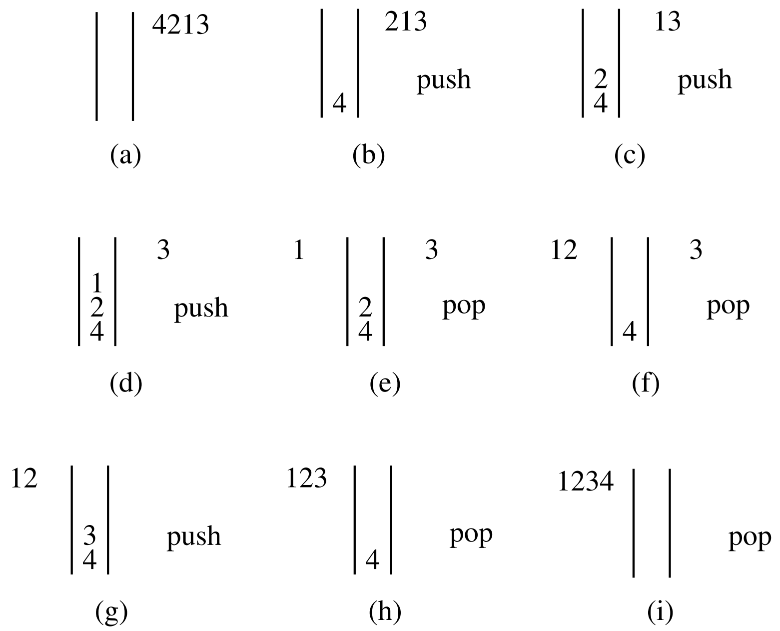


Figure 4: A sequence of operations sorting the permutation 4213 by a stack.

infinite basis and consists of five types of patterns. The enumeration of this class of patterns is still an open problem. We give the full description of the class below.

Theorem 1.30 (Pratt [123])**.**

$$S_n(\mathbb{D}\mathbb{e}\mathbb{q}) = Av(T_{\mathbb{D}}),$$

where $T_{\mathbb{D}}$ is the union of all the patterns in one of the following forms and any $k \geq 1$:

i) $(4k+1)2(4k-1)4\cdots(4k-2)(4k-5)(4k)(4k-3)$,

   obtained from the identity $12\cdots(4k+1)$, after leaving the even elements fixed, rotating the odd elements cyclically right by two places, and then interchanging $(4k+1)$ and $(4k-1)$.

ii) $(4k+1)2(4k+3)4\cdots(4k)(4k-3)(4k+2)(4k-1)$,

obtained from the identity $12\cdots(4k+3)$, after leaving the even elements fixed and rotating the odd elements cyclically right by two places.

iii) patterns in form i) or form ii), with the first two elements interchanged.

iv) patterns in form i), with the largest two elements, $4k$ and $4k+1$, interchanged or patterns in the form ii), with the largest two elements, $4k+2$ and $4k+3$, interchanged.

v) patterns in form i) or form ii), with both the largest two elements and the first two elements interchanged.

Thus, the set $T_{\mathbb{D}}$ contains four patterns of any odd size $l \geq 5$:

$$T_{\mathbb{D}} = \{52341, 25341, 42351, 24351, 5274163, 2574163, 5264173, 2654173, \ldots\}$$

The only permutation that can be sorted by a queue is the identity permutation and this, as Knuth writes [104, Chapter 2.2.1], follows trivially "by the nature of the queue".

Theorem 1.31.

$$S_n(\mathbb{Q}) = \{1, 12, 123, \ldots\} = Av(21).$$

1.3.2 Sorting by modifications of stacks

The results of Knuth on stack-sorting were followed by a great number of articles investigating sorting by different variations of a stack or networks of stacks. Some examples are *stacks in series*

[141, 144], *stacks in parallel* [63, 138], *pop-stacks* [12] and *stacks of bounded size* [10]. A few of the results in these works are discussed below.

1.3.2.1 Stacks in series

One can use a device several times in a row by using the output after one iteration as an input to the next iteration over the same device. This is the so-called *sorting in series*. Let us denote by $W_{n,k}$ the set of permutations in S_n, sortable with k iterations over a stack.

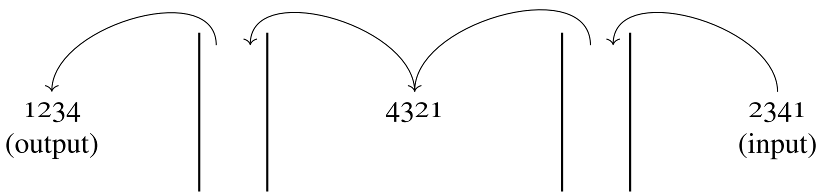


Figure 5: Sorting the permutation 2341 with 2 stacks in series.

We have $W_{n,1} = Av_n(231)$ and thus $|W_{n,1}| = \frac{\binom{2n}{n}}{n+1}$, the n-th Catalan number [108]. The set $W_{n,2}$ was obtained in the thesis of Julian West [141].

Theorem 1.32 (West [141])**.** A permutation $\pi \in W_{n,2}$ if and only if $\pi \in Av(2341)$, and if it does not have an occurrence of 3241, except possibly as part of an occurrence of 35241.

Giving a characterization for the sets $W_{n,t}$, when $t \geq 3$, is a complicated problem. Such characterization for the set of 3-stack sortable permutations, $W_{n,3}$, which requires a new type of permutation patterns, was given in [140].

The following formula for $|W_{n,2}|$ was first conjectured in the same thesis of West [141] and later proved by Zeilberger [144] with the help of computer.

Theorem 1.33 (Zeilberger [144])**.**

$$|W_{n,2}| = \frac{2(3n)!}{(n+1)!(2n+1)!}.$$

There is no known formula for $|W_{n,t}|$, when $t \geq 3$. Some recent progress on the asymptotics of $|W_{n,3}|$ was made by Defant [44] and by Bóna [28].

1.3.2.2 Stacks in parallel

Which permutations can be sorted if we use several stacks in parallel? This problem is significantly harder compared to the case of sorting by a single stack. Figure 6 illustrates how the permutation 4231, which cannot be sorted by a single stack, can be sorted by two stacks in parallel.

Both Tarjan [138] and Even & Itai [63] showed independently that no finite set of forbidden classical patterns can characterize the set of permutations sortable by k stacks in parallel, for $k \geq 2$. However, in 2015, Albert and Bousquet-Mélou enumerated the set of permutations that one can sort by 2 stacks in parallel. They found a pair of functional equations that characterise the corresponding generating function. Below, we formulate another important and recent result relating sorting by 2 stacks in parallel and sorting by a deque.

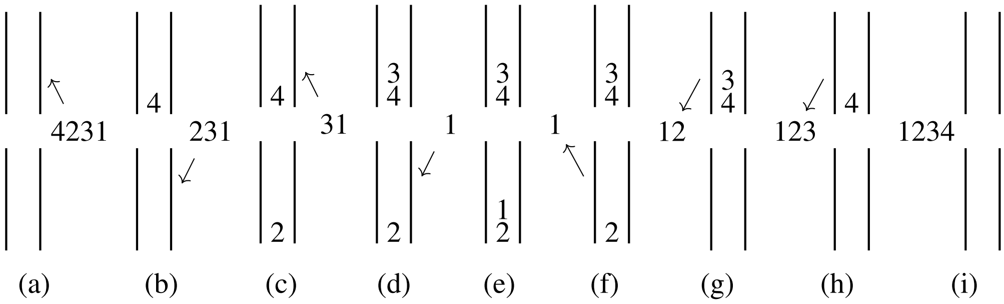


Figure 6: Sorting the permutation 4231 with 2 stacks in parallel.

Theorem 1.34 (Price [124])**.** If $S_n(\mathbb{SIP}_2)$ is the set of permutations in S_n that can be sorted by 2 stacks in parallel, then let $\mu_p := \lim_{n\to\infty} \sqrt[n]{|S_n(\mathbb{SIP}_2)|}$ and let $\mu_d := \sqrt[n]{\lim_{n\to\infty} |S_n(\mathbb{D}\mathbb{e}\mathbb{q})|}$. Then, μ_p and μ_d exist and

$$\mu_p = \mu_d.$$

1.3.2.3 Pop-stacks

A pop-stack is a stack, for which every pop operation unloads the entire content of the stack. Figure 7 below shows how a permutation can be sorted by a pop-stack.

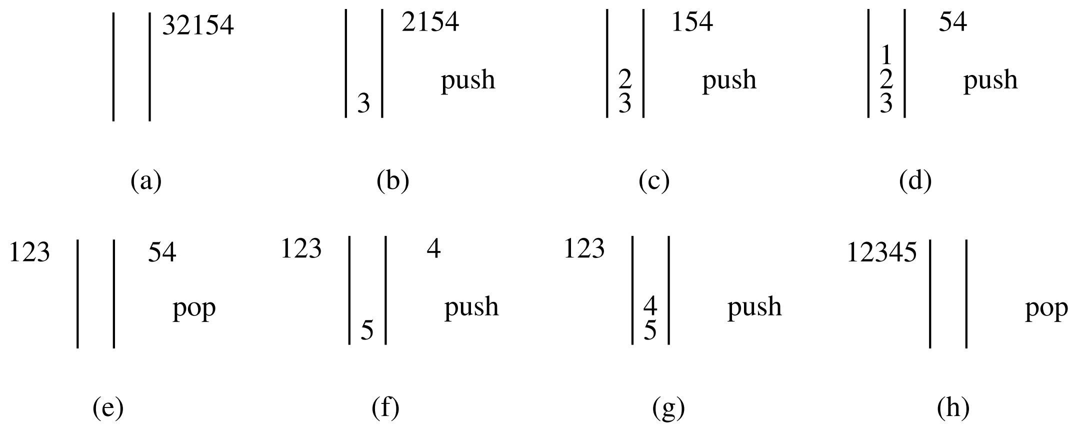


Figure 7: A sequence of operations sorting the permutation 32154 by a pop-stack.

Theorem 1.29 implies that a permutation cannot be sorted by a pop-stack if it contains the pattern 231. The following result shows that the avoidance of only one other pattern is necessary and sufficient for a permutation to be pop-stack sortable.

Theorem 1.35 (Avis and Newborn [12])**.** The set of permutations sortable by pop-stack is given by $Av(231, 312)$.

Alternatively, it is easy to show that a permutation can be sorted by a pop-stack if and only if it is one of the so-called *layered* permutations. These are the permutations consisting of disjoint union of factors (layers), such that the element decrease within each layer and increase between the layers. For example, 3215476 is a layered permutation with layers 321, 54 and 76. Using this additional fact, Avis and Newborn enumerated the class $Av_n(231, 312)$.

Theorem 1.36 (Avis and Newborn [12])**.**

$$av_n(231, 312) = 2^{n-1}.$$

Sorting by pop-stacks in series and by pop-stacks in parallel were investigated in [12] and [11], respectively. These works show that the sets of permutations sortable with a fixed number of pop-stacks in series (or in parallel) is, again, characterized by a finite set of forbidden classical patterns.

1.3.3 Sorting by modifications of deque and queue

The exercises at the end of the chapter defining stack-sorting in the book of Knuth [104, Chapter 2.2.1] consider sorting by two modifications of the device $\mathbb{Deq}$. Let $\mathbb{Deq}^{ir}$ denote an *input-restricted deque* and let $\mathbb{Deq}^{or}$ denote an *output-restricted deque*. These two devices are deques, which can perform an input and output operations, respectively, at only one of its ends. The possible operations for them are shown in Figure 8 below.

(a) Input-restricted deque (b) Output-restricted deque

Figure 8: The possible operations for input-restricted and output-restricted deque.

It is known that the permutations sortable by these two devices are, again, characterised by permutation classes.

Theorem 1.37 (Knuth [104])**.** $S(\mathbb{D}\mathrm{eq}^{ir}) = Av(3241, 4231)$.

Theorem 1.38 (West [141])**.** $S(\mathbb{D}\mathrm{eq}^{or}) = Av(2431, 4231)$.

One of the few articles discussing sorting by modification of a queue is [54], where Peter Doyle looks at a queue that is capable of doing direct transfers of elements from the input to the output. This is a reasonable modification since allowing direct transfers does not change which permutations can be obtained with the use of a stack. The same article establishes the following result.

Theorem 1.39 (Doyle [54])**.** If $\mathbb{Q}^{tr}$ is a queue, which can perform direct transfers to its output, then

$$S(\mathbb{Q}^{tr}) = Av(321).$$

A generalization of these type of devices, performing direct transfers of elements are considered in a work of Albert et. al [5].

1.4 Shuffling methods

Several articles [6, 8, 51, 139] investigate *shuffling methods* for a given deck of cards (not to be confused with "deque") or a given permutation. A shuffling method is a procedure that is comprised of the following two steps: choose a permutation out of a given set according to a given distribution. Then, apply this permutation over the deck. Any shuffling method has a *permutation family*, $F_\Sigma := \{\Pi_\Sigma^n \subseteq S_n \mid n = 2, \ldots\}$, which contains one set of size-n permutations, Π_Σ^n that can be applied when using the method, for every $n \geq 2$. An example of a shuffling method is *shuffling by cuts*, whose permutation family is given below. This method is studied in Chapter 2.

Definition 1.40. The permutation family of the shuffling method *cuts* is

$$F_{\text{cuts}} := \{\{k(k+1)\cdots n12\cdots(k-1) \mid k \in [2,n]\} \mid n \geq 2\}. \tag{1.1}$$

When one uses shuffling by cuts over an input of size *n*, one picks a permutation in the set $\Pi^n_{\text{cuts}} = \{k(k+1)\cdots n12\cdots(k-1) \mid k \in [2,n]\}$ according to uniform distribution and applies it to the input. The goal when applying any given shuffling method multiple times is to obtain a uniformly shuffled deck.

Diaconis, Fulman and Holmes [48, Section 2.3] give an overview of the previous work related to shuffling. The mathematics of shuffling uses tools related to mixing times [47], representation theory [73] and quasi-symmetric polynomials [133].

1.4.1 In-shuffles and Monge shuffles

In this section, we define the *In-shuffle* and *Monge* shuffling methods, as some of our results in Chapter 2 are related to them. The book of Diaconis and Graham [49, Chapter 6] discusses these and other shuffling methods.

The *In-shuffle* method is one of the two kinds of perfect riffle shuffles that are probably the most popular shuffling methods. When using the perfect riffle shuffles, half of the deck is held in each hand with the thumbs inward, then cards are released by the thumbs so that they fall to the table interleaved perfectly, i.e., the first card is coming from one of the halves, the second from the other half and so on. The Out-shuffles leaves the original top card back on top. The In-shuffles leaves the original top card second from top. For example, a deck of eight cards numbered by $1, 2, 3, 4, 5, 6, 7, 8$ from top to bottom, is transformed to $5, 1, 6, 2, 7, 3, 8, 4$ after one In-shuffle. Applications of the riffle shuffles and some of their mathematical properties are discussed in [9, 50]. The permutation family of the In-shuffle method is

$$\forall n \geq 2 : \Pi^n_{\text{In-sh}} = \begin{cases} \{(k+1)1(k+2)2\cdots(2k)k\}, \text{if } n = 2k, \text{ and} \\ \\ \{(k+1)1(k+2)2\cdots(2k)k(2k+1)\}, \text{if } n = 2k+1. \end{cases}$$

The *Monge shuffle* method is named after the eighteenth-century geometer Gaspard Monge, who worked out the basic mathematical details of these shuffles [49]. The Monge shuffle is carried out by successively putting cards over and under. The top card is taken into the other hand, the next is placed above, the third below these two cards and so on. For example, a deck of eight cards numbered by $1, 2, 3, 4, 5, 6, 7, 8$ from top to bottom, is transformed to $8, 6, 4, 2, 1, 3, 5, 7$ after one Monge shuffle. The permutation family of the Monge shuffling method is

$$\forall n \geq 2 : \Pi^n_{\text{Monge}} = \{\cdots 642135 \cdots\}.$$

1.5 Permutation statistics that are linear combinations of patterns

Our goal in this section is to define a large family of statistics on permutations, in order to prove certain facts about this family in Chapter 3. Let $A(\pi)$ be the set of distinct pairs of integers (u, v), such that u occurs before v in π. Formally,

$$A(\pi) := \{(u, v) \mid u = \pi_i, v = \pi_j, i < j\}.$$

We will need the following Definition 1.41 of *bivincular patterns* that will be used only in Chapter 3. In the same chapter, the bivincular patterns will be simply called "patterns." We use this non-standard definition, in order to be consistent with the notation in two previous works [37, 96], containing results, analogous to those in Chapter 3, for patterns in set partitions and for patterns in matchings, respectively.

Bivincular patterns generalize the concept of vincular patterns. They are vincular patterns, for which one is also allowed to require the numbers in every pattern occurrence, corresponding to some letters with consecutive values in the pattern, to have consecutive values in the permutation. Bivincular patterns were introduced by Bousquet-Mélou et al. in [32]. A main motivation for the authors was to find a minimal superset for the set of vincular patterns, which is closed under the inverse operation, in addition to the reverse and complement operations [102, Section 1.4].

Definition 1.41.

(i) A *permutation pattern* $\underline{P}$ of size k is a tuple $\underline{P} = (P, \boldsymbol{C}(\underline{P}), \boldsymbol{D}(\underline{P}))$, where $P = p_1 \cdots p_k$ is a permutation of size k and $\boldsymbol{C}(\underline{P}) \subseteq [k-1]$, $\boldsymbol{D}(\underline{P}) \subseteq [k-1]$ are two subsets.

(ii) An *occurrence* of the pattern $\underline{P} = (p_1 p_2 \cdots p_k, \boldsymbol{C}(\underline{P}), \boldsymbol{D}(\underline{P}))$ of size k in $\sigma \in S_n$ is a tuple $s = (t_1, t_2, \ldots, t_k)$ with $t_i \in [n]$, such that:

a) $t_1 < t_2 < \cdots < t_k$.

b) $(t_i, t_j) \in A(\sigma)$, if and only if $(i, j) \in A(P)$.

c) if $i \in \boldsymbol{C}(\underline{P})$, then $\sigma^{-1}(t_{p_{i+1}}) = \sigma^{-1}(t_{p_i}) + 1$, i.e., the positions of t_{p_i} and $t_{p_{i+1}}$ in σ are consecutive.

d) if $i \in \boldsymbol{D}(\underline{P})$, then $t_{i+1} = t_i + 1$, i.e., the values of t_i and t_{i+1} in σ are consecutive.

We will write $s \in_{\underline{P}} \sigma$ if s is an occurrence of $\underline{P}$ in σ. Note that an occurrence of a classical or vincular pattern in a permutation is a subsequence of this permutation, while an occurrence of a bivincular pattern is an increasing sequence of numbers in such subsequence.

Example 1.42. $\underline{P} = (4312, \{2\}, \{3\})$.
$t = (1, 3, 5, 6) \in_{\underline{P}} 625143$, since $\mathrm{red}(6513) = 4312$, the positions of $t_3 = 5$ and $t_1 = 1$ are consecutive, and $t_4 = 6 = t_3 + 1$.

Note that when $\boldsymbol{D}(\underline{P}) = \emptyset$, then $\underline{P}$ is a vincular pattern and that when both $\boldsymbol{C}(\underline{P}) = \emptyset$ and $\boldsymbol{D}(\underline{P}) = \emptyset$, then $\underline{P}$ is a classical pattern. In these two cases, we will not write $\underline{P}$ as a tuple, but we will use the notation introduced in Section 1.1 and Section 1.2. For instance, we will write $\underline{23}1$ instead of $(231, \{1\}, \emptyset)$, and 3124 instead of $(3124, \emptyset, \emptyset)$.

The number of occurrences of the pattern $\underline{P}$ in σ will be denoted by $\mathrm{cnt}_{\underline{P}}(\sigma)$. In the literature, usually a *permutation statistic* is a function $T : S \to \mathbb{N}$, where $S = \bigcup_{i=1}^{\infty} S_n$. In this thesis, when we write *statistic* or *simple statistic*, we will refer to two classes of such functions defined below. In Chapter 3, we will show how one can find the higher moments of any statistic in these two classes.

Definition 1.43. (i) A simple statistic $f_{\underline{P},Q}$ is defined by a pattern $\underline{P}$ of size k and a valuation function $Q(s, w) = Q_1(s)Q_2(w)$, which is a product of two polynomials $Q_1, Q_2 \in \mathbb{Z}[y_1, \ldots, y_k, m]$. If $\sigma \in S_n$ and $s = (t_1, t_2, \ldots, t_k) \in_{\underline{P}} \sigma$, such that $\sigma(w_i) = t_i$, for all $i \in [k]$, then write $Q(s, \sigma^{-1}(s)) = Q_1(s)Q_2(\sigma^{-1}(s)) = Q_1 \mid_{y_i = t_i, m = n} Q_2 \mid_{y_i = w_i, m = n}$. Let

$$f(\sigma) = f_{\underline{P},Q}(\sigma) := \sum_{s \in_{\underline{P}} \sigma} Q(s, \sigma^{-1}(s)) = \sum_{s \in_{\underline{P}} \sigma} Q_1(s)Q_2(\sigma^{-1}(s)).$$

Let the degree of a simple statistic $f_{\underline{P},Q}$, denoted $d(f)$, be the sum of twice the size of P and the degree of Q, which is the sum of the degrees of Q_1 and Q_2.

(ii) A *statistic* is a finite Q-linear combination of simple statistics. The degree of a statistic is defined to be the minimum, over all such representations, of the maximum degree of any of the included simple statistics.

Example 1.44. a) $\mathrm{cnt}_{\underline{P}} := f_{\underline{P},1}(\sigma) = \sum_{s\in_{\underline{P}}\sigma} 1$, which counts the number of occurrences of the pattern $\underline{P}$ in σ, is a simple statistic for any bivincular pattern $\underline{P}$, with valuation function $Q = 1$. This includes any classical and any vincular pattern. If the corresponding permutation P is of size k, then the degree of this simple statistic is $d(\mathrm{cnt}_{\underline{P}}) = 2k$. For instance, let us consider *the number of occurrences of* $\underline{P^*} = (312, \{2\}, \{2\})$:

$$\mathrm{cnt}_{\underline{P^*}}(\sigma) = \sum_{s\in_{\underline{P^*}}\sigma} 1.$$

It was shown in [61] that the number of permutations in S_n with k occurrences of this pattern is equal to the number of matchings on $[2n]$ with k right nestings and no left nestings.

b) *Descent drop.*

$$\mathrm{drops}(\sigma) = \sum_{\sigma_i>\sigma_{i+1}} \sigma_i - \sigma_{i+1} = \sum_{(t_1,t_2)\in_{\underline{21}}\sigma} t_2 - t_1$$

is a simple statistic corresponding to the pattern $(21, \{1\}, \emptyset)$ with valuation function $Q(s, w) = Q_1(s)Q_2(w)$, where $Q_1(s) = Q_1(t_1, t_2) = t_2 - t_1$ and $Q_2(w) = 1$. Thus, $deg(Q) = 1$ and $d(\mathrm{drops}) = 5$. Petersen and Tenner [121] showed that this statistic is equidistributed with the statistic $dp(\sigma) = \sum_{\sigma(i)>i} \sigma(i) - i$, which they call “depth.” The depth of a permutation is half of another important statistic called “total displacement” or “Spearman’s disarray,” whose generating function was found in [120].

c) *Sum of pinnacle squares.*

$$\text{pncSqSum}(\sigma) = \sum_{\sigma(i-1)<\sigma(i)>\sigma(i+1)} \sigma(i)^2 = \sum_{(t_1,t_2,t_3)\in_{\underline{132}}\sigma} t_3^2 + \sum_{(t_1,t_2,t_3)\in_{\underline{231}}\sigma} t_3^2$$

is a statistic, which is a sum of the two simple statistics $f_1 = f_{\underline{132},t_3^2}$ and $f_2 = f_{\underline{231},t_3^2}$. Thus, $d(\text{pncSqSum}) = max(d(f_1), d(f_2)) = 8$. A pinnacle in a permutation σ is a value $\sigma(i)$, such that $\sigma(i-1) < \sigma(i) > \sigma(i+1)$. An article investigating the number of permutations with a predefined set of pinnacles, called a “pinnacle set”, is [43]. To the best of our knowledge, the sum of the pinnacles and the sum of the squares of the pinnacles have not been yet investigated, despite the recent interest in pinnacle sets [52, 53, 127]. Note that the number of pinnacles and the sum of the peaks in a permutation are also statistics. We consider the sum of the squares for the pinnacles in a permutation to demonstrate the power of the methods we use. We find a closed form expressions for the first and the second moment of this more complicated statistic in Chapter 3.

1.5.1 Previous work on statistics in other combinatorial structures

Chern, Diaconis, Kane and Rhoades [37] considered a class of set partition statistics analogous to the class of permutation statistics given by Definition 1.43. They showed that if a set partition statistic belongs to this class, i.e., if it can be written as a linear combination of patterns, then its moments can be expressed as a linear combination of shifted Bell numbers with coefficients that are polynomials in n. Here, we state their result in a formal way.

Definition 1.45 (set partition)**.** A *set partition* of $[n]$ is a family of non-empty subsets of $[n]$, such that each of its elements is included in exactly one subset.

Example 1.46. $\{\{1, 4\}, \{2, 5\}, \{3\}\}$ is a partition of $[5]$.

Consider the set, Π_n, of all set partitions of $[n]$. Their number is B_n, the n-th Bell number [118, A000110].

Example 1.47. When $n = 3$, the set Π_3 contains $B_3 = 5$ set partitions of $[3]$:

$$\Pi_3 = \{\{1\}, \{2\}, \{3\}\}, \{\{1, 2\}, \{3\}\}, \{\{1, 3\}, \{2\}\}, \{\{1\}, \{2, 3\}\}, \{1, 2, 3\}.$$

Definition 1.48. Let f be a function defined over the set of all combinatorial structures of a given kind (e,g., set partitions, permutations, compositions, etc.). Denote the set of these structures, of size n, by C_n. Then, for a fixed n and $r \geq 0$, *the aggregate* of f^r is defined as

$$M(f^r; n) := \sum_{\lambda \in C_n} f(\lambda)^r.$$

Note that the *r-th moment* of f, that is, the expectation of f^r is given by

$$\mathbb{E}(f^r) = \frac{M(f^r; n)}{|C_n|}.$$

Example 1.49. Let $cr_2(\lambda)$ denotes the number of 2-crossings in the set partition λ, i.e., the number of tuples $i_1 < i_2 < j_1 < j_2$, such that i_1, j_1 and i_2, j_2 are in two different blocks. For instance, $cr_2(\{\{1, 3\}, \{2, 4, 5\}\}) = 2$.

It is known that the aggregate of cr_2 is given by the following formula (Kasraoui, [95]):

$$M(cr_2;n) = \sum_{\lambda \in \Pi_n} cr_2(\lambda) = \frac{1}{4}(-5B_{n+2} + (2n+9)B_{n+1} + (2n+1)B_n).$$

The approach of Chern et al. gives us a way to find similar formulas for the aggregates of cr_2^r for higher values of r, because this statistics counts the number of occurrences of a simple pattern in set partitions. For instance, they found the formula below:

$$M(cr_2^2;n) == \sum_{\lambda \in \Pi_n} cr_2(\lambda)^2 = \frac{1}{144}(225B_{n+4} - (180n+814)B_{n+3} + (36n^2+156n+489)B_{n+2}+$$

$$(72n^2+72n-260)B_{n+1} + (36n^2+24n-23)B_n).$$

The number of 2-crossings is just one of the many popular set partition statistics, for which similar formulas exist.

Theorem 1.50 (Chern et al. [37])**.** For each set partition statistics f, which is a linear combination of patterns in set partitions and for each $r \in \{0, 1, \ldots\}$, there exists a closed form expression

$$M(f^r;n) = P_{k,2k}(n)B_{n+2k} + P_{k,2k-1}(n)B_{n+2k-1} + \cdots + P_{k,0}(n)B_n,$$

where each $P_{k,2k-j}$ is a polynomial with rational coefficients. Moreover, the degree of $P_{k,2k-j}$ is

$$\begin{cases} j, & \text{if } j \leq k \\ k - \lceil \frac{j-k}{2} \rceil, & \text{if } j > k \end{cases}.$$

Khare, Lorentz and Yan [96] developed the approach of Chern et al. on the set of perfect matchings (set partitions with blocks of size 2). Let $\mathcal{M}_{2m}$ denotes the set of all perfect matchings on $[2m]$ and let $T_{2m} := |\mathcal{M}_{2m}| = (2m-1)(2m-3)\cdots 3 \cdot 1 = (2m-1)!!$ be called double factorials.

Theorem 1.51 (Khare et al. [96])**.** The aggregates of f^r, where f is a perfect matchings statistic that can be represented as a linear combination of patterns, can be written as linear combinations of double factorials with constant coefficients.

For example, the last theorem gives

$$\sum_{M \in \mathcal{M}_{2m}} cr_2^2(M) = \binom{2m}{4} T_{2m-4} + 12\binom{2m}{6} T_{2m-6} + 70\binom{2m}{8} T_{2m-8}. \tag{1.2}$$

Since the terms of the kind $\binom{2m}{k}$ are polynomials of $2m$ of degree k, we get that the right-hand side of Equation (1.2) can be written as a combination of double factorials with constant coefficients. Our goal in Chapter 3 will be to develop the same approach for statistics on permutations and to obtain similar closed form expressions for the aggregates (and respectively for the moments) of each permutation statistics in the class given by Definition 1.43.

1.5.2 Previous work on central limit theorems for permutation patterns

Some of our other results in Chapter 3 are related to giving new proofs of central limit theorems for permutation patterns. We give the necessary background in the current section, as well as a list of some past related results.

Definition 1.52 (normally distributed random variable)**.** A random variable $X = N(\mu, \sigma^2)$ is *normally distributed* with expectation μ and variance σ^2, if for any real number x,

$$\mathbb{P}(X \leq x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} e^{-(\frac{t-\mu}{\sigma})^2/2} dt.$$

Definition 1.53 (convergence in distribution)**.** A sequence $X_1, X_2, \ldots$ of real-valued random variables is said to *converge in distribution* to a random variable X if

$$\lim_{n \to \infty} F_n(x) = F(x),$$

for every number $x \in \mathbb{R}$ at which F is continuous. Here F_n and F are the cumulative distribution functions of the random variables X_n and X, respectively. We will write $X_n \underset{d}{\to} X$.

The normal distribution appears frequently in the context of combinatorial enumeration [27, Chapter 3]. A major reason is the central limit theorem, which gives us that under rather general circumstances, when independent random variables are added, their properly normalized sum converges in distribution to the normal distribution. In this case, we say that this variable is *asymptotically normal*.

Consider $X_n := \mathrm{cnt}_{\underline{P}}(\sigma)$ as a random variable, where $\sigma \in S_n$ is chosen uniformly at random and $\underline{P}$ is a fixed pattern. We do not have a straightforward way to write X_n as a sum of independent random variables and apply the central limit theorem, in order to obtain that $\mathrm{cnt}_{\underline{P}}$ is asymptotically normal. However, several previous works established the asymptotic normality of $\mathrm{cnt}_{\underline{P}}$ for different fixed patterns $\underline{P}$ or for an arbitrary $\underline{P}$ in a given set of patterns. For example, see Feller [65, 3rd ed., p.257] (for inversions, i.e., when $\underline{P} = 21$), Mann [109] (for descents, i.e., when $\underline{P} = \underline{21}$), Fulman [76] (for both inversions and descents), Goldstein [80] and Borga [29] (for consecutive patterns), Bóna [24] (for classical patterns) and Hofer [84] (for vincular patterns). However, the number of occurrences of some simple bivincular patterns is not normally distributed (see Section 3.4.3).

The recent works of Gaetz and Ryba [77] and Kammoun [93] establish normal limit laws on certain classes of permutations for classical and vincular patterns, respectively. In addition, Janson [90, 91] showed that the number of pattern occurrences is not normally distributed when we sample from the permutations avoiding a certain fixed pattern. Earlier, Janson, Nakamura and Zeilberger [92] initiated the study of the same general question. Two articles proving asymptotic normality for random permutations selected not according to the uniform measure are [41, 66]. Finally, some important works [16, 64, 92] give central-limit theorems for certain joint-distributions of pattern occurrences. The listed articles use various approaches, from the method of moments [146] to dependency graphs, Stein's method (see [84, Section 3] for overview of the last two methods) and the theory of U-statistics [89, Chapter XI].

In Section 3.4.1, we give a new proof of a lemma of Bóna, which was shown in [24] to imply the asymptotic normality of $\mathrm{cnt}_{\underline{P}}$ for any classical pattern $\underline{P}$. To do that, he uses dependency graphs and the so-called Janson dependency criterion, which are defined below.

Definition 1.54 (dependency graph)**.** Let $\{Y_{n,k} \mid k = 1, 2, \dots, N_n\}$ be an array of random variables. A graph G is a *dependency graph* for $\{Y_{n,k} \mid k = 1, 2, \dots, N_n\}$ if the following two conditions are satisfied:

1. There exists a bijection between the variables $Y_{n,k}$ and the vertices of G.
2. If V_1 and V_2 are two disjoint sets of vertices of G, so that there are no edges of G between V_1 and V_2, then the two sets of random variables corresponding to V_1 and V_2 are independent.

The idea of the method of dependency graphs is that if the degrees of the vertices in any sequence of dependency graphs, for a given family of variables, do not grow too fast, then the corresponding variables behave as if independent and their sum is asymptotically normal [68]. Janson's criterion gives one sufficient condition for this asymptotic normality, quantifying that the degrees do not grow too quickly.

Theorem 1.55 (Janson [88])**.** Let $\{Y_{n,k} \mid k = 1, 2, \dots, N_n\}$ be an array of random variables, such that for all $n \geq 1$ and for all $k = 1, 2, \dots, N_n$, the inequality $|Y_{n,k}| \leq A_n$ holds for some real number A_n, and the maximum degree of a dependency graph for $\{Y_{n,k} \mid k = 1, 2, \dots, N_n\}$ is Δ_n.

Set $Y_n := \sum_{i=1}^{k} Y_{n,k}$ and $\sigma_n^2 := Var(\sigma_n)$. If there is a natural number m, so that

$$N_n \Delta_n^{m-1} (\frac{A_n}{\sigma_n})^m \xrightarrow[n\to\infty]{} 0,$$

then

$$Y_n \xrightarrow[d]{} N(0, 1).$$

In Section 3.4.2, we give a new interpretation of a lemma of Hofer [84] that is a generalisation of the lemma of Bóna in [24]. Hofer used her lemma to establish the asymptotic normality of $\mathrm{cnt}_{\underline{P}}$ for any vincular pattern $\underline{P}$. To do that, she also used Lemma 1.57 stated below. Before we formulate it, we need to define the so-called Kolmogorov distance, which is a probability metric.

Definition 1.56 (Kolmogorov distance)**.** Let F_X and F_Y be the cumulative distribution functions of the real-valued random variables X and Y. The *Kolmogorov distance* between X and Y is defined as

$$d(X, Y) := \sum_{t \in \mathbb{R}} |F_X(t) - F_Y(t)|.$$

Lemma 1.57. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of real-valued random variables and let Y be another real-valued random variable. Then,

$$d(X_n, Y) \xrightarrow[n\to\infty]{} N(0,1) \quad \Longrightarrow \quad X_n \underset{d}{\to} 0.$$

1.6 Distant patterns

In this section, we introduce another type of permutation patterns, which are investigated in Chapter 5. *Distant patterns* (DPs) generalize the concept of classical patterns in a different way than the vincular patterns introduced in Section 1.2 (Definition 1.25). While vincular patterns allow one to require no gap between the numbers in the permutation, corresponding to two consecutive letters of the pattern, DPs allow arbitrary minimum requirements for the size of this gap. We will write $\square^r$ to denote a gap with at least r letters, with $\square := \square^1$.

Definition 1.58 (distant patterns)**.**

(i) A distant pattern of size k is a classical pattern σ of size k, for which we can have a symbol $\square^r$, for some $r \geq 1$, at each of the following $k+1$ places: between each of the $k-1$ pairs of consecutive letters of σ, before its first letter or after its last letter.

(ii) An occurrence of the distant pattern σ in the permutation π is an occurrence $\pi_{\lambda_1}\pi_{\lambda_2}\cdots\pi_{\lambda_k}$ of the corresponding classical pattern in π, such that if $\sigma_i\square^r\sigma_{i+1}$ is a segment in σ, then $\lambda_{i+1} - \lambda_i > r$.

Example 1.59. $12\square^2 3$ is a distant pattern. 257 is an occurrence of $12\square^2 3$ in the permutation 6253147 since $\mathrm{red}(257) = 123$ and we have more than 2 numbers between 5 and 7 in this permutation. Meanwhile, 234 is not an occurrence of $12\square^2 3$.

Any distant pattern can be written in the form

$$\square^{r_0}q_1\square^{r_1}q_2\square^{r_2}\cdots\square^{r_{k-1}}q_k\square^{r_k},$$

where each r_i is a non-negative integer and $q_1q_2\cdots q_k \in S_k$. We will also consider tight constraints and we will underline the corresponding part of the pattern in case of a tight constraint as, for example, in $\underline{1\square^4 2}3$ to denote that we want to avoid the pattern 123 with gap size exactly 4 between the letters 1 and 2. DPs without any tight constraints will be called *classical distant patterns*, while DPs having at least one tight constraint will be called *vincular distant patterns*. When all of the constraints for the gap sizes in a distant pattern are tight, we will call this a *consecutive distant pattern*.

Definition 1.60 (uniform distant patterns)**.** Take a classical pattern q and require the minimal gap size to be the same number r for all pairs of consecutive letters. We will denote this pattern by $\mathrm{dist}_r(q)$ and we will call these *uniform distant patterns*.

Example 1.61. $\mathrm{dist}_3(312) = 3\square^3 1\square^3 2$.

Note that DPs generalize classical patterns since $q = \mathrm{dist}_0(q)$ for any classical pattern q. Furthermore, one can write any vincular pattern as a vincular distant pattern and thus vincular distant patterns generalize vincular patterns. Finally, when we say that a distant pattern has size n, we mean that the number of its non-square letters is n. For example, $21\square^4 3$ is a distant pattern of size 3.

1.6.1 Previous work related to distant patterns

The idea of arbitrary constraints for the gap sizes between any two consecutive pattern letters is not new, even though not much has been written on the subject. The thesis of Ghassan Firro [70] defines a notion of permutation patterns with gap constraints, which is more general than DPs, and unifies many other popular pattern notations. He calls these patterns *distanced patterns* or *d-patterns*. The distanced patterns described there also allow requiring a gap size to be at most some given number r. The thesis itself enumerates the patterns of the kind $xy\square z$ using both a direct bijection and an analytical approach. We have included this result in Section 4.3.1. The paper of Hopkins and Weiler [85] describes the concept of uniform distant patterns under the name of *gap patterns* and obtains an important result related to them, as a corollary of their work on pattern avoidance over posets. We state this corollary in Section 4.3.2.

In his book dedicated to pattern avoidance [102], Kitaev discusses a few articles related to patterns containing the $\square$-symbol and he uses this symbol in the same way as we do. One such work mentioned in the book is by Hou and Mansour on the so-called Horse permutations [86]. There, the authors proved that the permutations avoiding both the classical pattern 132 and the pattern $1\square 23$ are in one–to–one correspondence with the so-called Horse paths.

In [98], Kitaev introduces *partially ordered patterns* (*POPs*) and *partially ordered generalized patterns* (*POGPs*) which further generalize classical DPs (respectively, vincular DPs). While in classical patterns, all of the letters form one totally ordered set (e.g. in 123, $1 < 2 < 3$), in POPs this set is partially ordered. In an occurrence of a distant pattern, any element at the place of a $\square$ is incomparable to any other element, which shows us that POPs (and POGPs that allow tight constraints) are indeed generalizations. If we have a classical distant pattern or a vincular distant pattern, we could easily write it as a POP (respectively, POGP) by replacing each square with a letter in its own group. POPs were studied in the context of permutations, words and compositions in a series of papers [83, 97–102] including a recent work [78] of Gao and Kitaev where a systematic search of connections between sequences in the Online Encyclopedia of Integer Sequences (OEIS) and permutations avoiding POPs of size 4 and 5 was conducted. The work of Claesson [38], generalizing previous results of Callan [34], studies avoidance of non-consecutive occurrence of a pattern, and this has connections with both POPs and DPs. Another generalization of the DPs are the so-called *place-difference-value patterns* [103].

1.7 Summary of the new results in the thesis

In Chapter 2, we consider sorting by special types of queues, called *shuffle queues*, that can rearrange their content by applying permutations corresponding to different shuffling methods. We obtain that sorting by a queue that can reverse its content is equivalent to sorting by a deque. We also show that the set of permutations that can be sorted by cuts, if the queue must be unloaded after a permutation is applied, is the set of the 321-avoiding separable permutations. Generalization of this fact is also obtained. Next, in Section 2.4, we investigate sorting by the same queue in series. The chapter continues with results on *pop shuffle queues*, which are shuffle queues that are unloaded by each pop operation. Finally, we formulate and investigate an astonishing conjecture, which states that one can sort the same number of permutations of a given size by using the pop shuffle queues for the well-known *In-shuffle* and *Monge* shuffling methods, defined in Section 1.4.1.

In Chapter 3, we study the class of statistics on permutations defined in Section 1.5 and their higher moments. After we adapt the approach of Chern, Diaconis, Kane and Rhoades to permutations, we show that the moments of any statistic in this class is a linear combination of factorials with constant coefficients. Using a corollary of this result, we obtain a new proof of the central limit theorem (CLT) for the number of occurrences of classical patterns, which uses a lemma of Burstein and Hästö. We give a simple interpretation of this lemma and an analogous lemma that would imply the CLT for the number of occurrences of any vincular pattern. Furthermore, we obtain explicit formulas for the r-th moment of the descents and the minimal descents statistics. The latter is used to give a new direct proof of the fact that we do not necessarily have asymptotic normality in the case of bivincular patterns.

Chapter 4 deals with avoidance of distant patterns (DPs), defined in Section 1.6. We describe a bijection between the permutations avoiding inversions with elements at distance more than a given number r and the permutations of size n for which any two elements in a cycle differ by at most r. We also sketch an approach to obtain the generating function for the number of permutations avoiding one of the two not yet classified DPs of size three and constraint on each gap size not exceeding one. The approach is based on the block-decomposition method initiated in [115]. Thereafter, we deduce a surprising relation between the sets of permutations avoiding the classical patterns 123 and 132 by looking at a class of DPs with tight gap constraints. Furthermore, we show how one can use DPs to give combinatorial proofs to two conjectures of Kuszmaul. We also investigate some analogues of the Stanley-Wilf former conjecture for DPs.

Chapter 5 lists some important further questions related to the presented results.

CHAPTER 2

SORTING BY SHUFFLING METHODS AND A QUEUE

In this chapter, we relate sorting devices and shuffling methods by considering sorting using special type of queues, called *shuffle queues*, which can rearrange their content by applying permutations from a given predefined set over it. We will consider collections of permutations corresponding to some popular shuffling methods.

2.1 Definitions

A shuffling method Σ is defined by a family of sets of permutations

$$F_\Sigma = \{\Pi^n_\Sigma \subseteq S_n \mid n = 2, \dots\}$$

that one can apply over the content of a sorting device when using the method. Note that Π^n_Σ contains permutations of size n, for every $n \geq 2$. We will also assume that $id_n \notin \Pi^n_\Sigma$, for every $n \geq 2$ and we will refer to F_Σ as the *permutation family* of the method Σ. We will also use the notation $(\Pi^k_\Sigma)^{-1} := \{\sigma^{-1} \mid \sigma \in \Pi^k_\Sigma\}$. An example of a shuffling method is shuffling by cuts defined in Section 1.4.

For a given shuffling method Σ, we consider a non-deterministic sorting device $\mathbb{Q}_\Sigma$ for which at any given step one can apply up to three possible operations over the current configuration $s = (s_{inp}, s_{dev}, s_{out})$. Denote the next configuration by $\overline{s}$. The three operations are described below.

1. **Push:** Move the first element x of the input $s_{inp} = xs'_{inp}$ to the content of the device. We get $\overline{s} = (s'_{inp}, s_{dev}x, s_{out})$. One can apply this operation only if $s_{inp} \neq \varepsilon$.

2. **Pop:** Move the first element y of the content of the device $s_{dev} = ys'_{dev}$ to the output. We get $\overline{s} = (s_{inp}, s'_{dev}, s_{out}y)$. One can apply this operation only if $s_{dev} \neq \varepsilon$.

3. **Shuffle:** Choose a permutation $\sigma \in \Pi^m_\Sigma$ and apply it over the content of the device s_{dev}, where $|s_{dev}| = m$. We get $\overline{s} = (s_{inp}, \sigma s_{dev}, s_{out})$. One can apply this operation only if $m \geq 2$ and if the last operation that has been applied is not a shuffle operation.

Note that the device $\mathbb{Q}_\Sigma$ functions as a queue since it can receive entries on one of its ends and release entries on the other end. In addition, the content of this queue can be shuffled and thus we will call it a *shuffle queue*. When a certain permutation is chosen to be applied on a shuffle operation, we will say that the shuffle operation is associated with this permutation.

Also, note that the restriction to not have two consecutive shuffle operations is reasonable since if one allows applying multiple consecutive shuffle operations for a shuffle queue $\mathbb{Q}_\Sigma$, then sorting by this queue would be equivalent to sorting by a queue $\mathbb{Q}_{\Sigma'}$, for which two consecutive shuffle operations are not allowed. Here, Σ' would be the shuffling method for which $\Pi^n_{\Sigma'} = \langle \Pi^n_\Sigma \rangle$, for every $n \geq 2$, where $\langle T \rangle$ denotes the subgroup generated by the set T.

2.1.1 Devices of type (i) and type (ii)

The current chapter focuses on two natural variations of the devices $\mathbb{Q}_\Sigma$ that will be called shuffle queues of type (i) and type (ii). They are obtained after imposing two additional restrictions.

(i) The entire content of the device must be unloaded after each shuffle.

Denote the corresponding sorting device by $\mathbb{Q}'_\Sigma$.

(ii) The entire content of the device must be unloaded by each pop operation.

Denote the corresponding sorting device by $\mathbb{Q}_{\Sigma}^{\mathsf{pop}}$. This is the pop-version of the device $\mathbb{Q}_{\Sigma}$ in analogy to the pop version of the stack-sorting device first considered by Avis and Newborn in [12]. We will also call them *pop shuffle queues*.

Consider the device of type (i), $\mathbb{Q}'_{\text{cuts}}$. Example 2.1 given below shows one possible sequence of configurations for $\mathbb{Q}'_{\text{cuts}}$ and the corresponding operations when sorting the permutation 213564 with this device. Recall that we call this *iteration* of $\mathbb{Q}'_{\text{cuts}}$ over 213564. Each configuration $(s_{inp}, s_{dev}, s_{out})$ is written as a column.

Example 2.1. Iteration of $\mathbb{Q}'_{\text{cuts}}$ over 213645.

$$\begin{pmatrix} 213645 \\ \varepsilon \\ \varepsilon \end{pmatrix} \xrightarrow{\mathsf{push}} \begin{pmatrix} 13645 \\ 2 \\ \varepsilon \end{pmatrix} \xrightarrow{\mathsf{push}} \begin{pmatrix} 3645 \\ 21 \\ \varepsilon \end{pmatrix} \xrightarrow[\text{+unload}]{\substack{\mathsf{shuffle} \\ (cut)}} \begin{pmatrix} 3645 \\ \varepsilon \\ 12 \end{pmatrix} \xrightarrow{\mathsf{push}} \begin{pmatrix} 645 \\ 3 \\ 12 \end{pmatrix}$$

$$\xrightarrow{\mathsf{pop}} \begin{pmatrix} 645 \\ \varepsilon \\ 123 \end{pmatrix} \xrightarrow{\mathsf{push}} \begin{pmatrix} 45 \\ 6 \\ 123 \end{pmatrix} \xrightarrow{\mathsf{push}} \begin{pmatrix} 5 \\ 64 \\ 123 \end{pmatrix} \xrightarrow{\mathsf{push}} \begin{pmatrix} \varepsilon \\ 645 \\ 123 \end{pmatrix} \xrightarrow[\text{+unload}]{\substack{\mathsf{shuffle} \\ (cut)}} \begin{pmatrix} \varepsilon \\ \varepsilon \\ 123456 \end{pmatrix}$$

This device requires that we unload the entire content of the device after each shuffle operation. Also, note that this is a non-deterministic device and one can choose to apply one among several different permutations on each shuffle operation. Consider the device of type (ii), $\mathbb{Q}_{\text{cuts}}^{\mathsf{pop}}$. Below is shown one possible iteration of $\mathbb{Q}_{\text{cuts}}^{\mathsf{pop}}$.

Example 2.2. Iteration of $\mathbb{Q}_{\text{cuts}}^{\text{pop}}$ over 41325.

$$\begin{pmatrix} 41325 \\ \varepsilon \\ \varepsilon \end{pmatrix} \xrightarrow{\text{push}} \begin{pmatrix} 1325 \\ 4 \\ \varepsilon \end{pmatrix} \xrightarrow{\text{push}} \begin{pmatrix} 325 \\ 41 \\ \varepsilon \end{pmatrix} \xrightarrow{\text{push}} \begin{pmatrix} 25 \\ 413 \\ \varepsilon \end{pmatrix} \xrightarrow[\text{}]{\substack{\text{shuffle} \\ (cut)}} \begin{pmatrix} 25 \\ 341 \\ \varepsilon \end{pmatrix} \xrightarrow{\text{push}} \begin{pmatrix} 5 \\ 3412 \\ \varepsilon \end{pmatrix}$$

$$\xrightarrow{\substack{\text{shuffle} \\ (cut)}} \begin{pmatrix} 5 \\ 1234 \\ \varepsilon \end{pmatrix} \xrightarrow{\substack{\text{pop} \\ (unload)}} \begin{pmatrix} 5 \\ \varepsilon \\ 1234 \end{pmatrix} \xrightarrow{\text{push}} \begin{pmatrix} \varepsilon \\ 5 \\ 1234 \end{pmatrix} \xrightarrow{\substack{\text{pop} \\ (unload)}} \begin{pmatrix} \varepsilon \\ \varepsilon \\ 12345 \end{pmatrix}$$

The device in Example 2.2 requires that we unload the entire content of the device by each pop operation, but we do not have to do that after a shuffle operation.

2.1.2 Motivation behind shuffle queues and cut-sorting

Here, we describe some additional motivation to consider sorting by shuffle queues, as well as their variations of types (i) and (ii). We also motivate the investigation of sorting by cuts, which is a main focus of the present work.

In Section 2.2, we show that sorting by a deque is equivalent to sorting by a simple shuffle queue. Perhaps, one could find shuffle queues that mirror sorting by other popular devices. This would give new perspectives and might help solve certain problems related to these devices. In addition, sorting by $\mathbb{Q}_{\text{cuts}}$ has a simple interpretation in terms of railway switching networks, which was the way used by Knuth in [104] to illustrate sorting by stack, queue and deque.

Example 2.3. (Knuth's railroad interpretation for $\mathbb{Q}_{\text{cuts}}$) Add a circular railroad extension connecting the beginning and the end of a railroad queue, as in Figure 9 below.

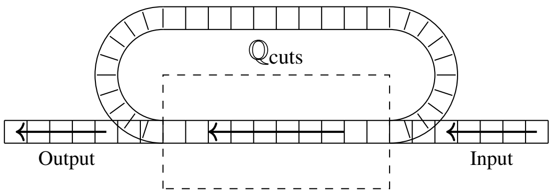


Figure 9: The shuffle queue $\mathbb{Q}_{\text{cuts}}$ represented as a railway switching network.

Suppose that a railroad car cannot enter or leave the queue (no pushes or pops are allowed), while there is a car in the extension. Thus we have a queue that can move a group of consecutive elements from its beginning to its end. This is exactly what one can do by cuts.

We also show that one can sort every permutation using $\mathbb{Q}_{\text{cuts}}$ (Corollary 2.22 gives an even stronger statement). Thus, it is reasonable to ask which permutations can be sorted by cuts and by other methods if we consider the two natural restrictions defining shuffle queues of types (i) and (ii), namely, to unload the content after each shuffle or with each pop, respectively. Sorting by the shuffle queue of type (i), $\mathbb{Q}'_{\text{cuts}}$ corresponds to sorting by the railway switching network shown at Figure 9, with the additional requirement that we have to unload the queue after each use of the extension.

The device $\mathbb{Q}'_{\text{cuts}}$ is non-deterministic and we show that by using this device one can sort a subset of the separable permutations defined in Section 1.1.2. Therefore, there exists a deterministic procedure that sorts all of the $\mathbb{Q}'_{\text{cuts}}$-sortable permutations in linear time, since we have such a procedure for the separable permutations [30]. This is something desirable when considering sorting devices on a restricted class of permutations since the best possible time complexity for a sorting algorithm over all permutations is $O(n \log n)$. The popular greedy stack-sorting gives such a linear deterministic procedure for stack. The PhD thesis of Luca Ferrari [69, Section 3.4] shows that such a procedure exists for input-restricted and output-restricted deques, and does not exist for a standard deque.

Sorting by cuts turns out to be an important problem connected to genome rearrangements and an object of extensive study from the algorithms community. For more details, we refer to the introduction of [82]. In particular, if we have two permutations representing sequences of genes, we want to find the shortest sequence of operations in a given set that transforms one of the permutations into the other. Assuming that one of the permutations is the identity, the problem is to find the shortest way of sorting a permutation using the fixed set of operations, e.g., cuts and others. The article of Eriksson et al. [60] is one work motivated by genome rearrangements that contains results on sorting by cuts which are closest to the bounds we obtain in Theorems 2.14 and 2.18. They establish bounds for the maximum number of cuts one must apply when sorting a permutation, while we give bounds for the maximum number of iterations of $\mathbb{Q}'_{\text{cuts}}$ needed to sort a permutation. The two problems are different, since during an iteration one can apply multiple cuts. Several other articles addressing sorting by cuts together with additional operations, e.g. possible reversions, are listed in [42].

Finally, considering sorting by shuffle queues of type (ii) (pop-shuffle queues) is reasonable since pop-sorting has been sufficiently considered in the past (see [102, Chapter 2.1.4]). Pop-shuffle queues require a natural additional constraint and we prove some interesting results involving them. In addition, we formulate a surprising conjecture relating the shuffle queues of type (ii) for two very different shuffling methods (see Section 2.5.2).

2.2 Shuffle queues equivalent to deque and stack

As we explained in Section 2.1.2, one motivation to consider shuffle queues is that sorting by deque turns out to be equivalent to sorting by the shuffle queue of a very simple shuffling method that can just reverse its content.

Definition 2.4. (the shuffling method rev)

$$\forall n \geq 2 : \Pi^n_{\text{rev}} = \{n(n-1)\cdots 21\}.$$

For a sequence w, the reverse of w will be denoted by w^r, as in Chapter 1.

Definition 2.5. Sorting devices $\mathbb{U}$ and $\mathbb{V}$ are *equivalent* if, for every $n \geq 1$,

$$S_n(\mathbb{U}) = S_n(\mathbb{V}).$$

We denote this by $\mathbb{U} \cong \mathbb{V}$.

Theorem 2.6. $\mathbb{D}\mathrm{eq} \cong \mathbb{Q}_{\text{rev}}$.

Proof. [First part: $S_n(\mathbb{D}\mathrm{eq}) \subseteq S_n(\mathbb{Q}_{\text{rev}})$] Let $\pi \in S_n(\mathbb{D}\mathrm{eq})$. Then, there exists an iteration of $\mathbb{D}\mathrm{eq}$ over π that sorts it. Take one such iteration itr, determined by a sequence of the operations $I, O, \overline{I}$ and $\overline{O}$. Using this sequence, we can easily construct an iteration of $\mathbb{Q}_{\text{rev}}$ that sorts π, as follows. Instead of the operation $\mathsf{shuffle}$ over $\mathbb{Q}_{\text{rev}}$, we will write $\mathsf{reverse}$. Replace I by push, O by pop, $\overline{I}$ by $\mathsf{reverse}, \mathsf{push}, \mathsf{reverse}$

and $\overline{O}$ by $\mathsf{reverse}, \mathsf{pop}, \mathsf{reverse}$. This yields a list of operations defining an iteration over $\mathbb{Q}_{\text{rev}}$, which modifies π in the exact same way as itr has modified π over $\mathbb{Deq}$.

[Second part: $S_n(\mathbb{Q}_{\text{rev}}) \subseteq S_n(\mathbb{Deq})$] If s is a sequence of operations over $\mathbb{Deq}$, then denote by $\overline{s}$ the *complement* sequence obtained by swapping $I \leftrightarrow \overline{I}$ and $O \leftrightarrow \overline{O}$. Take $\pi \in S_n(\mathbb{Q}_{\text{rev}})$ and a sequence of operations s corresponding to an iteration of $\mathbb{Q}_{\text{rev}}$ that sorts π. The sequence s consists of push, pop and reverse operations. Replace every push by an I and every pop by an O to obtain a sequence s'. Then, for each reverse operation in s', from left to right, replace the sequence of operations to its left by its complement sequence and then delete that reverse operation. We claim that you will obtain a sequence of operations s'' for the device $\mathbb{Deq}$ that sorts π. For example, suppose that

$$s = \mathsf{push}, \mathsf{push}, \mathsf{reverse}, \mathsf{pop}, \mathsf{reverse}, \mathsf{push}, \mathsf{pop}, \mathsf{push}, \mathsf{reverse}, \mathsf{pop}, \mathsf{pop}.$$

Then,

$$s' = I, I, \mathsf{reverse}, O, \mathsf{reverse}, I, O, I, \mathsf{reverse}, O, O.$$

We have three reverse operations in s'. If we follow the described procedure, we get:

$$s' \rightsquigarrow \overline{I}, \overline{I}, O, \mathsf{reverse}, I, O, I, \mathsf{reverse}, O, O$$

$$\rightsquigarrow I, I, \overline{O}, I, O, I, \mathsf{reverse}, O, O$$

$$\rightsquigarrow \overline{I}, \overline{I}, O, \overline{I}, \overline{O}, \overline{I}, O, O =: s''.$$

We will show that the iteration over $\mathbb{Deq}$ corresponding to s'' always sorts π. Assume that s (respectively, s') has r reverse operations denoted by rev_i (respectively, rev'_i), for $i \in [r]$. Furthermore, while transforming s' to s'', let the sequence of operations preceding rev_i, before replacing it with its complement sequence, be denoted by $s_{(i)}$, for $i \in [r]$. Note that its complement sequence is denoted by $s'_{(i)}$, for $i \in [r]$. Our goal is to prove that $s_{(i)}$ transforms π in the same way as $s'_{(i)}$, for $i \in [r]$.

We will proceed by induction. The sequence $s_{(1)}$ transforms π in the same way as $s'_{(1)}$, since $s_{(1)}$ is the complement of $s'_{(1)}$ with a reverse operation added at the end and it is easy to see that if s is a sequence of operations over $\mathbb{Deq}$ that produces output π' on input π, then $\overline{s}$ produces $(\pi')^r$ on input π. Therefore, if $s'_{(1)}$ produces output $\pi'_{(1)}$ on input π, then $s_{(1)}$ produces the same output $((\pi'_{(1)})^r)^r = \pi'_{(1)}$ on input π. Assume that the statement holds for all $i \leq t$ and that $t < r$. By the induction hypothesis, $s'_{(t)}$ transforms the input π in the same way as $s_{(t)}$. To obtain $s'_{(t+1)}$ and $s_{(t+1)}$, respectively from $s'_{(t)}$ and $s_{(t)}$, we should first add the same sequence of push and pop operations. Then we take the complement of $s'_{(t)}$ and add a reverse operation to $s_{(t)}$, respectively. We obtain the sequences $s'_{(t+1)}$ and $s_{(t+1)}$ that obviously transform the input π in the same way. If $t = r$, then we just add the same sequence of push and pop operations to $s'_{(t)}$ and $s_{(t)}$ to obtain s and s'', respectively. Therefore, these two sequences transform π in the same way and thus the iteration over $\mathbb{Deq}$ corresponding to s'' also sorts π. □

Once we know that Theorem 2.6 holds, a reasonable question to ask is whether there exists a shuffle queue that is equivalent to a stack. Recall that the device stack is denoted by $\mathbb{St}$.

Theorem 2.7. There is no shuffling method Σ, such that $\mathbb{St} \cong \mathbb{Q}_{\Sigma}$.

Proof. Suppose that such a shuffling method Σ exists. Then, we must have $S_n(\mathbb{Q}_\Sigma) = Av_n(231)$. Therefore, since $21 \in Av_2(231)$, we must have $21^{-1} = 21 \in \Pi^2_\Sigma$. We also have $231 \notin S_3(\mathbb{Q}_\Sigma)$. If $321 \in \Pi^3_\Sigma$, then we will be able to sort 231 by the following iteration:

$$\begin{pmatrix} 231 \\ \varepsilon \\ \varepsilon \end{pmatrix} \xrightarrow{\text{push}} \begin{pmatrix} 31 \\ 2 \\ \varepsilon \end{pmatrix} \xrightarrow{\text{push}} \begin{pmatrix} 1 \\ 23 \\ \varepsilon \end{pmatrix} \xrightarrow[\text{(by } 21)]{\text{shuffle}} \begin{pmatrix} 1 \\ 32 \\ \varepsilon \end{pmatrix}$$

$$\xrightarrow{\text{push}} \begin{pmatrix} \varepsilon \\ 321 \\ \varepsilon \end{pmatrix} \xrightarrow[\text{(by } 321)]{\text{shuffle}} \begin{pmatrix} \varepsilon \\ 123 \\ \varepsilon \end{pmatrix} \xrightarrow{\text{pop}} \begin{pmatrix} \varepsilon \\ \varepsilon \\ 123 \end{pmatrix}$$

Thus $321 \notin \Pi^3_\Sigma$. However, we must be able to sort 321 by $\mathbb{Q}_\Sigma$. Consider an input 321. In order to obtain 123, a pop operation must not be performed before the first three pushes. Note that after pushing the first two elements, one can either switch them or not, since $21 \in \Pi^2_\Sigma$. Therefore, after pushing the third element 1, one could either have 231 or 321 in the device. Thus we can sort 321 only if $321^{-1} = 321 \in \Pi^3_\Sigma$ or if $231^{-1} = 312 \in \Pi^3_\Sigma$. However, we saw that $321 \notin \Pi^3_\Sigma$. In addition, $312 \notin \Pi^3_\Sigma$, since otherwise we would be able to sort 231. This is a contradiction. □

In Section 5.1, we ask a more general question related to shuffle queues equivalent to devices that can sort all the permutations in a given permutation class.

2.3 Sorting by cuts

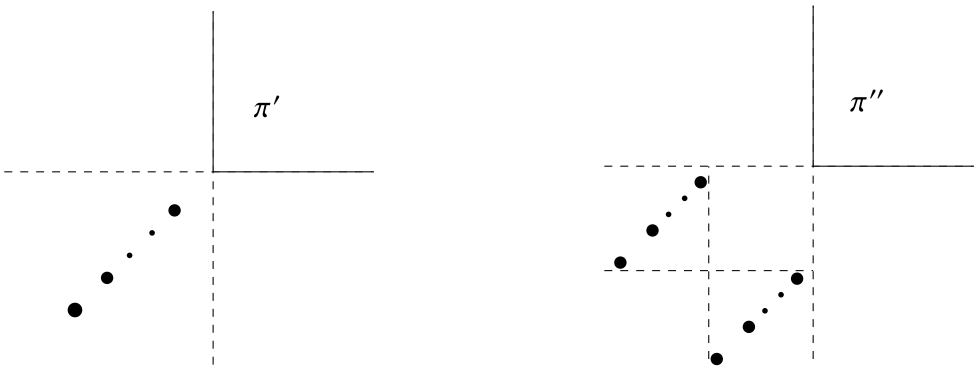
One of the simplest shuffling methods is shuffling by cuts. Its permutation family is given in Section 1.4. Some previous works containing results on shuffling using cuts are [46, 76]. The significance of sorting by $\mathbb{Q}_{\text{cuts}}$ and $\mathbb{Q}'_{\text{cuts}}$ is discussed in Section 2.1.2. Sorting by $\mathbb{Q}_{\text{cuts}}$ turns out to be trivial since one can sort every permutation with this shuffle queue. A more general statement is proved at the beginning of Section 2.5. In this section, we investigate sorting by $\mathbb{Q}'_{\text{cuts}}$. Example 2.1 in Section 2.1.1 shows one possible iteration of this device.

First, we determine $S_n(\mathbb{Q}'_{\text{cuts}})$, with the help of the following Lemma 2.8. We will call it the set of *cut-sortable permutations*. We obtain that this is the set of the separable permutations avoiding the pattern 321.

Lemma 2.8. A permutation $\pi \in S_n(\mathbb{Q}'_{\text{cuts}})$ if and only if it has one of the forms:

1. $\pi = id_r \oplus \pi'$, for some $1 \leq r \leq n$ and $\pi' \in S_{n-r}(\mathbb{Q}'_{\text{cuts}})$, or
2. $\pi = (id_{r_1} \ominus id_{r_2}) \oplus \pi''$, for some $r_1, r_2 \geq 1$, where $r := r_1 + r_2 \leq n$ and $\pi'' \in S_{n-r}(\mathbb{Q}'_{\text{cuts}})$.

Proof. Let $\pi = \pi_1 \cdots \pi_n \in S_n(\mathbb{Q}'_{\text{cuts}})$. Consider an iteration of $\mathbb{Q}'_{\text{cuts}}$ over π that sorts it. The sequence of operations for this iteration must contain at least one pop operation. Let the first pop operation be performed after we have pushed r elements in the device ($1 \leq r \leq n$), i.e., the elements $\pi_1, \ldots, \pi_r$. The output string after this pop operation must be id_r. We can have at most one shuffle operation before the first pop operation, and this shuffle must be right before the pop. If we do not have such a shuffle, then the content of the device has not been modified, i.e., $\pi_1 \cdots \pi_r = id_r$. Thus, $\pi = id_r \oplus \pi'$ and the rest of the iteration sorts π'. Therefore, $\pi' \in S_{n-r}(\mathbb{Q}'_{\text{cuts}})$. If a shuffle has been performed before the first pop, then before this shuffle, the device must contain one of the permutations in the set $(\Pi^r_{\text{cuts}})^{-1} = \Pi^r_{\text{cuts}}$. Each permutation in Π^r_{cuts} can be written as $id_{r_1} \ominus id_{r_2}$ for some $r_1, r_2 \geq 1$, such that $r := r_1 + r_2 \leq n$. Therefore, $\pi = (id_{r_1} \ominus id_{r_2}) \oplus \pi''$ for some permutation $\pi'' \in S_{n-r}(\mathbb{Q}'_{\text{cuts}})$ since π'' is sortable by the rest of the considered iteration. Conversely, one can directly check that any permutation in one of the two listed forms belongs to $S_n(\mathbb{Q}'_{\text{cuts}})$. □



(a) Cut-sortable permutations that require no shuffle before the first pop.

(b) Cut-sortable permutations that require a shuffle before the first pop.

Figure 10: The two kinds of cut-sortable permutations.

An equivalent formulation of Lemma 2.8 is that the set $S_n(\mathbb{Q}'_{\text{cuts}})$ consists of the permutations that can be obtained by direct sums of the trivial permutation 1 and permutations of the kind $id_{r_1} \ominus id_{r_2}$. The fact that $S_n(\mathbb{Q}'_{\text{cuts}})$ is a permutation class follows directly from a simpler version of the observation used to obtain Proposition 1 in [1]. With the next theorem, we find this class.

Theorem 2.9. The permutations sortable by $\mathbb{Q}'_{\text{cuts}}$ are the 321-avoiding separable permutations [118, A034943]; i.e.,

$$S_n(\mathbb{Q}'_{\text{cuts}}) = Av_n(321, 2413, 3142). \tag{2.1}$$

Proof. Let $T := \{321, 2413, 3142\}$.

[First part: π is cut-sortable $\Rightarrow \pi \in Av_n(T)$] We will use induction, Lemma 2.8 and the fact that if $\pi = x \oplus y$ for some permutations x, y and π has an occurrence of a pattern in T, then this occurrence is either in the part of π corresponding to x or in the part corresponding to y. This will be called the *indecomposable property* of T.

The empty permutation belongs to $Av_0(T)$. Let $n > 0$. Assume, inductively, that any cut-sortable permutation of size $m < n$ belongs to $Av_m(T)$. Suppose that $\pi \in S_n$ is cut-sortable and $\pi = id_r \oplus \pi'$, for some $1 \leq r \leq n$ and $\pi' \in S_{n-r}(\mathbb{Q}'_{\text{cuts}})$, as in the first form described in Lemma 2.8. Then $\pi' \in Av_{n-r}(T)$ by the inductive hypothesis and id_r has no occurrence of a pattern in T. Therefore, by the indecomposable property of T, we have $\pi \in Av_n(T)$.

Now suppose that π is in the second form described in the lemma, i.e., that $\pi = (id_{r_1} \ominus id_{r_2}) \oplus \pi''$, for some $r_1, r_2 \geq 1$, where $r := r_1 + r_2 \leq n$ and $\pi'' \in S_{n-r}(\mathbb{Q}'_{\text{cuts}})$. Then, $\pi'' \in Av_n(T)$ by the induction hypothesis and one can check easily that $id_{r_1} \ominus id_{r_2}$ has no occurrence of a pattern in T. Because of the indecomposable property of T, we must have $\pi \in Av_n(T)$.

[Second part: $\pi \in Av_n(T) \Rightarrow \pi$ is cut-sortable] We will use induction, again. The empty permutation is the only permutation in $Av_0(T)$, and it is cut-sortable. Let $n > 0$ and $\pi = \pi_1 \cdots \pi_n \in Av_n(T)$. Consider the consecutive segment $12 \cdots r$ in π for the greatest possible value of r, where

$\pi = \pi_1 \cdots \pi_l 12 \cdots r \pi_{r+l+1} \cdots \pi_n$. If $\pi_1 \cdots \pi_l$ is the empty permutation, then π has the first form from Lemma 2.8. If not, then $l \in [1, n-r+1]$ and we will show that π has the second form from the lemma.

First, note that $\pi_1 \cdots \pi_l$ must be increasing to avoid a 321 pattern in π. Assume that $\pi_1 \cdots \pi_l \neq (r+1)(r+2) \cdots (r+l)$ and let $u \geq 1$ be minimal, such that $\pi_u \neq r+u$. We must have that $u \in [1, l]$, $r+l+1 \leq n$, $r+u \in \{\pi_{r+l+1}, \cdots, \pi_n\}$ and $\pi_l > r+u$. If $u > 1$, then $\pi_1 = r+1$ and $(r+1)\pi_l 1(r+u)$ would form a 2413 pattern in π, which will contradict $\pi \in Av_n(T)$. Consider $u = 1$. Note that $\pi_{r+l+1} \neq r+1$ since r was maximal. In fact, $\pi_{r+l+1} > r+1$. If $\pi_{r+l+1} < \pi_l$, then $\pi_l \pi_{r+l+1}(r+1)$ would form a 321 pattern, while if $\pi_{r+l+1} > \pi_l$, then $\pi_l 1 \pi_{r+l+1}(r+1)$ would form a 3142 pattern. Therefore, we must have $\pi_1 \cdots \pi_l = (r+1)(r+2) \cdots (r+l)$ and thus π has the second form from Lemma 2.8. □

In [117], Martinez and Savage showed that $a_n := av_n(321, 2413, 3142)$ satisfies

$$a_n = 3a_{n-1} - 2a_{n-2} + a_{n-3},$$

with initial conditions $a_1 = 1$, $a_2 = 2$, $a_3 = 5$. This is sequence A034943 in the OEIS [118]. The recurrence implies that $a_n = \Theta(d^n)$, where the growth rate $d \approx 2.32$.

Next, we prove a generalisation of Theorem 2.9.

Definition 2.10. (irreducible permutation) An irreducible permutation $\pi \in S_n$ is one for which the first j elements, i.e., those in $[j]$, do not occupy the first j positions, for any $0 < j < n$.

Example 2.11. When $n = 3$, the only irreducible permutations are 231, 312 and 321 since they do not have 1, 12 or 21 as a prefix.

Denote by IRR_n the set of the irreducible permutations of size n. They are enumerated by sequence A003319 in [118].

Theorem 2.12. If $\Pi_\Sigma^k \subseteq IRR_k$ for every $k \geq 2$ and $b_k := |\Pi_\Sigma^k|$, then

$$p_n(\mathbb{Q}'_\Sigma) = 1 + \sum_{\substack{k_1+\cdots+k_l=n-u \\ k_i \geq 2, u \geq 0}} \binom{u+l}{l} \prod_{j=1}^{l} b_{k_j}. \tag{2.2}$$

Proof. Recall that a subsequence $\pi_a\pi_{a+1}\cdots\pi_b$ of π, for which the indices $a, a+1, \ldots, b$ are consecutive numbers is called a segment of π. We denote it by $[a, b]$. Note that when we use $\mathbb{Q}'_{\text{cuts}}$, the entire content has to be unloaded after each shuffle and the segments of the input that were not shuffled are kept the same in the output. Thus the output after an iteration of $\mathbb{Q}'_{\text{cuts}}$ is uniquely determined by the segments of the input that were shuffled and the corresponding permutations chosen for each of the shuffle operations. For instance, the output id_6 of the iteration of $\mathbb{Q}'_{\text{cuts}}$ shown in Example 2.1 is determined by the sequence of segments $([1, 2], [4, 6])$ of the input 213645 that were shuffled and the sequence of permutations $(21, 231)$ that were applied on the given segments. We will use that one can make the same observation for any $\mathbb{Q}'_\Sigma$ with the given properties.

Denote the set of the possible pairs of sequences of segments and permutations, for an input of size n and a shuffling method Σ, by SSP_n^Σ. For every $n \geq 2$ and every element $(s, q) \in \mathrm{SSP}_n^\Sigma$, the segments in s are in lexicographical order and do not overlap with each other since we shuffle these segments from left to right. We will first show that $|\,\mathrm{SSP}_n^\Sigma\,|$ is equal to the expression in the right-hand side of Equation (2.2). Then, we will give a bijection between the sets $S_n(\mathbb{Q}'_\Sigma)$ and SSP_n^Σ.

[Finding $|\,\mathrm{SSP}_n^{\Sigma}\,|$] Assume that $x = (s, q) \in \mathrm{SSP}_n^{\Sigma}$ and that s consists of l shuffled segments. Only one such x exists, if $l = 0$. Let $l \geq 1$. Denote the sizes of the l shuffled segments by $k_1, k_2, \ldots, k_l$, where $k_j \geq 2$ for every $j \in [1, l]$, and let their sum be $n - u$ for some $u \geq 0$. For instance, if $n = 8$, $l = 2$ and $s = ([2, 3], [5, 7])$, then $k_1 = 2$, $k_2 = 3$ and $n - u = 5$. In general, if the numbers $l, n - u, k_1, \ldots, k_l$ are given, then in order to determine the sequence of segments s, one should distribute the u remaining elements in the set of $l + 1$ spaces: one before each of the l segments and one after all of the segments. For every such choice, we obtain a different sequence of segments s. The number of these choices is the number of ways to distribute u indistinguishable balls into $l + 1$ boxes that is $\binom{u+(l+1)-1}{(l+1)-1} = \binom{u+l}{l}$. Then, if $q = (q_1, \ldots, q_l)$, the permutation q_j can be any of the b_{k_j} permutations in $\Pi_{\Sigma}^{k_j}$ for every $j \in [1, l]$. Thus q can be determined in $\prod_{j=1}^{l} b_{k_j}$ ways. In total, we obtain the right-hand side of Equation (2.2).

[$S_n(\mathbb{Q}'_{\Sigma}) \to \mathrm{SSP}_n^{\Sigma}$] Let $\pi \in S_n(\mathbb{Q}'_{\Sigma})$. Then, there exists at least one iteration that sorts π. Assume that π can be sorted by two different iterations it_1 and it_2, corresponding to $x_1, x_2 \in \mathrm{SSP}_n^{\Sigma}$, where $x_1 = (s_1, q_1)$, $x_2 = (s_2, q_2)$ and $x_1 \neq x_2$. Assume that $s_1 = s_2$. Then, $q_1 \neq q_2$. However, we can easily see that this is not possible. Let $[r, r + k]$ be an arbitrary segment in s_1, and respectively in s_2. If σ_1 and σ_2 are the two permutations in q_1 and q_2, respectively, that have to be applied on this segment, then we must have $\sigma_1 = \sigma_2 = (\pi_r \cdots \pi_{r+k})^{-1}$. Thus $q_1 = q_2$. Therefore, we must have $s_1 \neq s_2$.

Let $[r_1, r_1 + k_1]$ be the last segment in s_1 and let $[r_2, r_2 + k_2]$ be the last segment in s_2. Assume also that σ_1 and σ_2 are the last permutations in q_1 and q_2, respectively. We saw that if $[r_1, r_1 + k_1] = [r_2, r_2 + k_2]$, then we must have $\sigma_1 = \sigma_2$. However, $s_1 \neq s_2$. Therefore, without loss of generality, assume that $[r_1, r_1 + k_1] \neq [r_2, r_2 + k_2]$ and that $r_1 \leq r_2$. If $r_1 = r_2$, then assume for concreteness that $k_1 < k_2$ (see Figure 11).

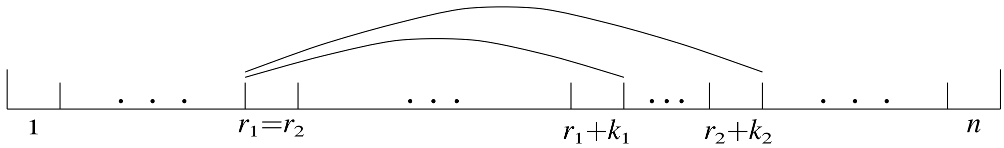


Figure 11: The case $r_1 = r_2$.

Iteration it_1 permutes the elements of the segment $[r_1, r_1 + k_1]$ in π. Hence it_2 does the same. This implies that σ_2 fixes $[k_1 + 1]$ and thus $\sigma_2 \notin IRR_{k_2+1}$, which is a contradiction. If $r_1 < r_2$, then it suffices to look at the following two cases (see Figure 12 and Figure 13).

1. $r_2 \leq r_1{+}k_1$. Then σ_1 fixes $[r_2 - r_1]$. Indeed, suppose that $\sigma_1(u) = v$, where $u \in [r_2 - r_1]$ and $v > r_2 - r_1$. This means that it_1 moves π_{r_1+u-1} to position $r_1 + v - 1 \geq r_1 + (r_2 - r_1 + 1) - 1 = r_2$. However, it_2 moves π_{r_1+u-1} to a position smaller than r_2. Therefore, σ_1 fixes $[r_2 - r_1]$ and σ_1 is not irreducible, which is a contradiction.

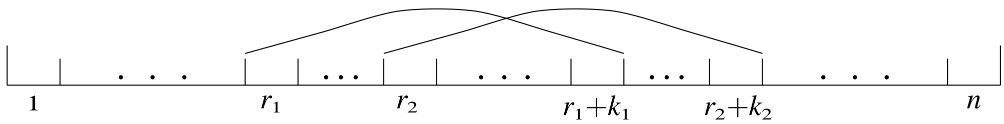


Figure 12: The case $r_1 < r_2$ and $r_2 \leq r_1 + k_1$.

2. $r_2 > r_1{+}k_1$. Since it_1 sorts π, we must have $\sigma_2 = id_{k_2+1}$, which is not possible.

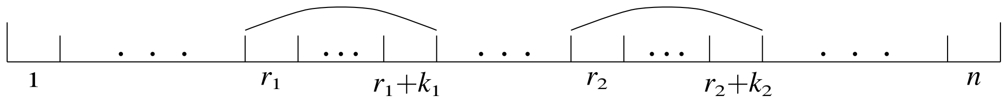


Figure 13: The case $r_1 < r_2$ and $r_2 > r_1 + k_1$.

We see that it is not possible to sort π by two different iterations corresponding to two different elements of SSP_n^{Σ}. Therefore, for every $\pi \in S_n(\mathbb{Q}'_{\Sigma})$ there exists a unique $x \in \mathrm{SSP}_n^{\Sigma}$ corresponding to an iteration that sorts π.

$[\mathrm{SSP}_n^{\Sigma} \to S_n(\mathbb{Q}'_{\Sigma})]$ It remains to show that every $x \in \mathrm{SSP}_n^{\Sigma}$ corresponds to a set of iterations of $\mathbb{Q}'_{\Sigma}$ sorting exactly one permutation π. Let $x = (s, q)$, where $s = (s_1, \ldots, s_l)$ and $q = (\sigma_1, \ldots, \sigma_l)$. Take id_n, and go backwards by applying consecutively σ_j^{-1} to the segment s_j, for $j = l, l-1, \ldots, 1$. We will obtain a unique permutation π that is sortable by any iteration corresponding to x. $\square$

Note that when $\Sigma = cuts$, we have $\Pi^k_{\mathrm{cuts}} \subseteq IRR_k$ and $b_k = |\Pi^k_{\mathrm{cuts}}| = k - 1$, for every $k \geq 2$. Thus, one can use Equation (2.2) to compute $p_n(\mathbb{Q}'_{\mathrm{cuts}})$.

2.4 Permutations of higher cost

Obviously, not all $\pi \in S_n$ are sortable by $\mathbb{Q}'_{\mathrm{cuts}}$. However, one can consider sorting in series for this device (see Section 1.3.2.1). Recall that when we sort in series, we use the output after one iteration over the device as an input to the next iteration. Denote the set of permutations that one can obtain after k iterations of $\mathbb{Q}'_{\mathrm{cuts}}$ over a permutation $\pi \in S_n$ by $(\mathbb{Q}'_{\mathrm{cuts}})^k(\pi)$.

Definition 2.13 (permutation cost)**.** The *cost* of π is the minimum number of iterations needed to sort π using the device $\mathbb{Q}'_{\text{cuts}}$; i.e.,

$$\text{cost}(\pi) := \min\{m \mid id_n \in (\mathbb{Q}'_{\text{cuts}})^m(\pi)\}.$$

It is not difficult to obtain an upper bound for cost(π). Indeed, one can move a single element to its correct position using only one iteration. In particular, if the input permutation is $\pi = 12\cdots(i-1)\pi_i\cdots\pi_{j-1}i\pi_{j+1}\cdots\pi_n$, then one can perform an iteration consisting of only one cut right before i, after getting the subsequence $\pi_i\cdots\pi_{j-1}i$ into the device. Such an iteration will move i to its correct position. Consecutive movements of $i, i+1, \ldots, n$ to their correct positions will sort the permutation. Therefore, cost$(\pi) \leq n$. This upper bound is improved significantly with the theorem given below.

Theorem 2.14. For every $\pi \in S_n$, we have cost$(\pi) \leq \lceil \frac{n}{2} \rceil$, for all $n \geq 1$.

Proof. A computer simulation shows that the statement is true for $1 \leq n \leq 10$. Let $n \geq 11$ and let us assume, inductively, that the statement holds for all $n' < n$. The main observation that will be used is that if we have $k+1$ consecutive numbers in $[n]$, forming a segment in π, then we can treat them as a single element and apply the induction hypothesis for $n-k$. Two more observations will be substantially used that describe cases when we can modify π with one iteration over $\mathbb{Q}'_{\text{cuts}}$ and then use the main observation above. A third observation for the case when π is a direct sum of other permutations will also be needed. These three observations are listed below with a brief justification for each of them:

(1) If $a \in [3, n]$ and the numbers $a-1, a-2$ occur before a in π, then there exists $\pi' \in \mathbb{Q}'_{\text{cuts}}(\pi)$, such that $\pi' = \cdots(a-2)(a-1)a\cdots$.

Proof: Assume that $\pi = \pi_1 \cdots (a-1) \cdots (a-2)\pi_h \cdots a \cdots \pi_n$ for some $h > 2$. If $(a-2)$ is before $(a-1)$, then we can proceed in a similar way. We can perform the following two cuts with one iteration of π over $\mathbb{Q}'_{\text{cuts}}$: Cut the segment $(a-1) \cdots (a-2)$ after $(a-1)$ and the segment $\pi_h \cdots a$ before a. A permutation π' with the desired property is obtained. If $a = \pi_h$, then we will not need the second cut.

(2) Assume that $a, a+1, b, b+1 \in [n]$ are four different numbers. If a and $a+1$ occur before b and $b+1$ in π, then there exists $\pi' \in \mathbb{Q}'_{\text{cuts}}(\pi)$, such that $\pi' = \cdots a(a+1) \cdots b(b+1) \cdots$

Proof: If necessary, we can move $a+1$ to appear immediately after a with a single cut, and $b+1$ to appear immediately after b with another cut. Since a and $a+1$ occur before b and $b+1$, we can perform the two cuts within one iteration.

(3) Assume that $\pi = \sigma_1 \oplus \sigma_2 \oplus \cdots \oplus \sigma_k$. Then,

$$\text{cost}(\pi) \leq \max\{\text{cost}(\sigma_1), \text{cost}(\sigma_2), \ldots, \text{cost}(\sigma_k)\}.$$

Proof: Within one iteration, one may independently transform each of the parts of π corresponding to $\sigma_1, \sigma_2, \ldots, \sigma_k$. Thus, if $m = \max\{\text{cost}(\sigma_1), \ldots, \text{cost}(\sigma_k)\}$, then π can be sorted with m iterations.

We continue with the proof. Let xy denote the last two elements of π. Observation (1) implies that unless $y = 1$ or $y = 2$, we will be able to transform π to a permutation π' containing the segment $(y-2)(y-1)y$ with just one iteration. Looking at this segment as a single element and applying the induction hypothesis for $n-2$ would give us $\text{cost}(\pi) \leq 1 + \lceil \frac{n-2}{2} \rceil = \lceil \frac{n}{2} \rceil$, which is what we want. Assume that $y = 1$. Obviously, if $x \neq 2, 3$, we will be able to apply observation (1) again with $a = x$ and obtain the bound via the same calculation. If $x = 2$, then since $n \geq 4$, the numbers 3 and 4 will precede 1 and 2 in π which allows us to use observation (2) and obtain the permutation π' described there, with one iteration of π over $\mathbb{Q}'_{\text{cuts}}$. Treating both $1, 2$ and $3, 4$ as a single element and applying the induction hypothesis gives us the same calculation and implies the desired result, again. Therefore, $xy = 31$ is the only case that remains to be considered if $y = 1$.

If $y = 2$, then if $x = 1$, we can move the last two elements $xy = 12$ to the beginning of the permutation with an iteration consisting of a single cut to obtain a permutation $\pi' = 12 \oplus \pi''$. Applying observation (3) to π' and the induction hypotheses for π'' gives $\text{cost}(\pi) \leq 1 + \text{cost}(\pi'') \leq 1 + \lceil \frac{n-2}{2} \rceil \leq \lceil \frac{n}{2} \rceil$. Therefore, we may assume that if $y = 2$, then $x \neq 1$. If $x \neq 3, 4$, then we can obtain the result using observation (1), as before. If $x = 3$, then since $n \geq 5$, we will be able to apply observation (2) for $2, 3$ and $4, 5$. Therefore, $xy = 42$ is the only case that remains to be considered if $y = 2$.

We saw that it suffices to look at those permutations π having last two elements, $xy = 31$ or $xy = 42$. Following the same reasoning, we can easily obtain that it suffices to only look at permutations π beginning either with $n(n-2)$ or $(n-1)(n-3)$. The only difference is that an observation analogous to (1) shall be used dealing with the cases when $a - 2$ precedes both $a - 1$ and a. Hence, we have four cases, in total. We will show how we can complete the proof in only one of them, namely when π begins with $n(n-2)$ and finishes with 42. The proofs in the other 3 cases can be completed following the same reasoning.

Let $\pi = n(n-2)\cdots 42$. Then, we can assume that $n-3$ and $n-1$ occur after both 1 and 3, because otherwise we will be able to apply observation (2) for certain pairs of elements. For concreteness, let us take 3 to be before 1 and $n-1$ to be before $n-3$. The following argument works regardless of this order. We can assume that $\pi = n(n-2)\cdots 3 \cdots 1 \cdots (n-1) \cdots (n-3) \cdots 42$. Below, we show a particular way to transform π by four iterations. We give the output at the end of each iteration. The reader may try to find the exact cuts applied in these iterations.

$$\begin{array}{c}
n(n-2)\cdots 3 \cdots 1 \cdots (n-1) \cdots (n-3) \cdots 42 \\
\downarrow \\
n(n-2)\cdots (n-1) \cdots (n-3) \cdots 3 \cdots 142 \\
\downarrow \\
\cdots (n-1)n(n-2)(n-3)\cdots 3142 \\
\downarrow \\
3142 \cdots (n-1)n(n-2)(n-3)\cdots \\
\downarrow \\
3142 \cdots (n-1)n(n-2)(n-3)
\end{array}$$

Therefore, with four iterations, π can be transformed to $\pi' = w_1\pi'' w_2$, where $|\pi''| = n-8$, $|w_1| = 4$, $|w_2| = 4$ and $\pi' = \mathrm{red}(w_1) \oplus \mathrm{red}(\pi'') \oplus \mathrm{red}(w_2)$. Recall that $n \geq 11$ and thus $|\pi''| = n - 8 \geq 3$, which means that $\mathrm{cost}(\pi'') \geq 2$. In addition, $\mathrm{cost}(\sigma) = 2$, for any $\sigma \in S_4$. Therefore, observation (3) applied over π' and the induction hypothesis for π'' gives us $\mathrm{cost}(\pi') \leq \mathrm{cost}(\pi'') \leq \lceil \frac{n-8}{2} \rceil$, which implies $\mathrm{cost}(\pi) \leq 4 + \mathrm{cost}(\pi') \leq 4 + \lceil \frac{n-8}{2} \rceil = \lceil \frac{n}{2} \rceil$. $\square$

Theorem 2.14 gives a tight upper bound for the cost function since there exist permutations in S_n having cost $\lceil \frac{n}{2} \rceil$. For instance, $\mathrm{cost}(83527461) = 4$. The best absolute lower bound is obviously 0 since $\mathrm{cost}(id_n) = 0$ for every n. Let $M(n) := \max_{\pi \in S_n} \mathrm{cost}(\pi)$ be the maximal cost of a permutation of size n. Theorem 2.14 gives us that $M(n) \leq \lceil \frac{n}{2} \rceil$. Next, we give a lower bound for $M(n)$ by Theorem 2.18. We begin by showing that cost is monotonically increasing with respect to pattern containment. Recall that $C_n(q)$ denotes the permutations in S_n that contain the pattern q. A main fact that will be used is that sorting by cuts has the property defined below.

Definition 2.15 (hereditary property)**.** A shuffling method Σ has the *hereditary* property if the following holds: Suppose that a sequence σ can be transformed into a sequence σ' by a permutation in $\bigcup_{n=2}^{\infty} \Pi_{\Sigma}^{n}$. If τ is a subsequence of σ and its symbols transform into the subsequence τ' of σ', then there exists a permutation in $\bigcup_{n=2}^{\infty} \Pi_{\Sigma}^{n}$ transforming τ into τ'.

This property is defined in [1], as a property of the so-called "permuting machines". Here we will use that shuffling by cuts has this property.

Lemma 2.16. If $\pi \in C_n(q)$, then $\mathrm{cost}(\pi) \geq \mathrm{cost}(q)$.

Proof. Let us fix an occurrence oc of q in π. Assume that we have a sequence of iterations sorting π and let itr be one of these iterations. Every cut c in itr is transforming a certain sequence of elements σ to a sequence σ'. If τ is the subsequence of σ, including all of the elements of oc, that is transformed to a sequence τ', then by the hereditary property of sorting by cuts, there exists a cut c' which transforms τ to τ'. Therefore, for every sequence of iterations of $\mathbb{Q}'_{\mathrm{cuts}}$ that sorts π, one can get a corresponding sequence of iterations that sorts its subsequence oc by substituting each cut c in an iteration from the initial sequence with the corresponding cut c'. The total number of iterations may drop since some of the iterations in the initial sequence sorting π may not affect the elements of oc. If we consider an optimal sequence of cost(π) iterations sorting π, then the described correspondence gives a sequence of at most cost(π) iterations of $\mathbb{Q}'_{\text{cuts}}$ sorting q. Thus cost$(q) \leq$ cost(π). □

Recall that id_n^r is the reverse identity: $id_n^r = n(n-1)\cdots 1$.

Lemma 2.17. If $\pi' \in \mathbb{Q}'_{\text{cuts}}(id_n^r)$, then $\pi' \in C_n(id^r_{\lceil \frac{n}{2} \rceil})$, i.e., π' contains a decreasing subsequence of size $\lceil \frac{n}{2} \rceil$.

Proof. We will proceed by induction. The lemma holds for $n = 1$. Consider the first cut c in an arbitrary iteration of $\mathbb{Q}'_{\text{cuts}}$ over id_n^r. Denote the output permutation after this iteration by $\pi' = \pi'_1 \cdots \pi'_n$. If n does not participate in c, then $\pi'_1 = n$, because we have just pushed and popped $\pi_1 = n$. Then we can look at the considered iteration as one over id^r_{n-1} with the element n appended in front of the output. The element n is in front of any decreasing subsequence in $\pi'_2 \cdots \pi'_n$. Therefore, we may apply the induction hypothesis to get that π' must have a decreasing subsequence of size $1 + \lceil \frac{n-1}{2} \rceil \geq \lceil \frac{n}{2} \rceil$.

If n participates in c, then let the cut c be performed after we have exactly $k \geq 2$ numbers in the device $\mathbb{Q}'_{\text{cuts}}$: $n(n-1)\cdots(n-k+1)$. After the cut c, these k numbers will be divided into two decreasing sequences. In other words, $\pi'_1 \cdots \pi'_k$ will be comprised of two segments that are decreasing sequences. At least one of these sequences must be of size at least $\lceil \frac{k}{2} \rceil$ and therefore $\pi'_1 \cdots \pi'_k$ contains a decreasing sequence of such size. The rest of the iteration can be looked at as an iteration over id^r_{n-k}. Thus we can apply the inductive hypothesis to see that $\pi'_{k+1} \cdots \pi'_n$ must contain a decreasing sequence of size $\lceil\frac{n-k}{2}\rceil$. In addition, $\pi'_1\cdots\pi'_k > \pi'_{k+1}\cdots\pi'_n$, so π' must contain a decreasing sequence of size $\lceil\frac{k}{2}\rceil + \lceil\frac{n-k}{2}\rceil \geq \lceil\frac{n}{2}\rceil$. □

Now, we are ready to establish a lower bound for $M(n)$, i.e., the maximal cost of a permutation of size n.

Theorem 2.18. $M(n) \geq \lceil\log_2 n\rceil$, for each $n \geq 2$.

Proof. We will prove that $\mathrm{cost}(id_n^r) \geq \lceil\log_2 n\rceil$ for each $n \geq 2$, using induction. Obviously, $id_2^r = 21$ cannot be sorted with less than one iteration through $\mathbb{Q}'_{\mathrm{cuts}}$. By Lemma 2.17, after one iteration over id_n^r, we will always get a permutation $\pi' \in C_n(id^r_{\lceil\frac{n}{2}\rceil})$. By Lemma 2.16 and the induction hypothesis,

$$\mathrm{cost}(\pi') \geq \mathrm{cost}(id^r_{\lceil\frac{n}{2}\rceil}) \geq \left\lceil \log_2 \left\lceil \frac{n}{2} \right\rceil \right\rceil \geq \left\lceil \log_2 \frac{n}{2} \right\rceil = \lceil\log_2 n\rceil - 1.$$

But, $\pi' \in \mathbb{Q}'_{\mathrm{cuts}}(id_n^r)$, so $\mathrm{cost}(id_n^r) = 1 + \mathrm{cost}(\pi') \geq 1 + (\lceil\log_2 n\rceil - 1) = \lceil\log_2 n\rceil$. □

There exist values of n for which $M(n) > \lceil\log_2 n\rceil$. A question concerning the limit of $M(n)$ is formulated in Section 5.1.

We finish this section by showing that the permutations in S_n can be paired up in terms of cost, when using $\mathbb{Q}'_{\mathrm{cuts}}$. Recall that for a permutation $\pi = \pi_1\cdots\pi_n$, $\overline{\pi}$ denotes the complement permutation, defined by $\overline{\pi}_i = n+1-\pi_i$ and that π^r denotes the reverse of π, i.e., $(\pi^r)_i = \pi_{n+1-i}$. Set $\pi^* = \overline{\pi^r} = (\overline{\pi})^r$. Observe also that Π^n_{cuts} is closed under the * operation; i.e., for all $\sigma \in \Pi^n_{\mathrm{cuts}}$, we have $\sigma^* \in \Pi^n_{\mathrm{cuts}}$. Indeed, $(k(k+1)\cdots n12\cdots(k-1))^* = (n+2-k)\cdots(n-1)n12\cdots(n+1-k) \in \Pi^n_{\mathrm{cuts}}$, for each $k \in [2,n]$ and $n \geq 2$.

Theorem 2.19. For any permutation π, $\mathrm{cost}(\pi) = \mathrm{cost}(\pi^*)$.

Proof. We will show that $\mathrm{cost}(\pi^*) \le \mathrm{cost}(\pi)$. The equality follows because $(\pi^*)^* = \pi$, which will imply that $\mathrm{cost}(\pi) \le \mathrm{cost}(\pi^*)$. Let $\pi = \pi_1 \cdots \pi_n$. Consider an arbitrary iteration itr over π, consisting of m cuts associated with the permutations $\sigma_1, \ldots, \sigma_m$, respectively. Let the cut σ_k be applied over the segment $[i_k, j_k]$ in π, for $k \in [m]$. Denote the output permutation after the iteration itr with π'. Consider an iteration itr^* over π^*, corresponding to itr, that also consists of m cuts, given by the permutations $\sigma_1^*, \ldots, \sigma_m^*$ which are applied over the segments $\pi^*_{n+1-j_k} \cdots \pi^*_{n+1-i_k}$, for $k \in [m]$. If $(\pi^*)'$ is the output permutation after applying itr^*, then we claim that $(\pi^*)' = (\pi')^*$. This implies that for any sequence of iterations $\mathsf{itr}_1, \mathsf{itr}_2, \ldots, \mathsf{itr}_r$ that sorts π, one would have a corresponding sequence of iterations $\mathsf{itr}_1^*, \mathsf{itr}_2^*, \ldots, \mathsf{itr}_r^*$ that sorts π^*, since $id_n^* = id_n$:

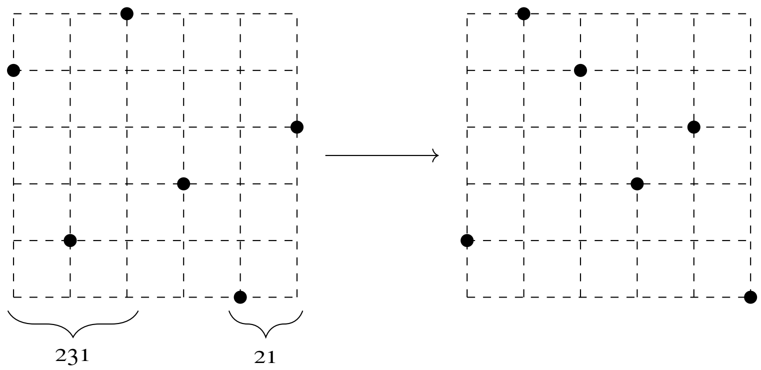
$$\pi \xrightarrow{\mathsf{itr}_1} \pi' \xrightarrow{\mathsf{itr}_2} \cdots \xrightarrow{\mathsf{itr}_r} id_n$$

$$\pi^* \xrightarrow{(\mathsf{itr}_1)^*} (\pi')^* \xrightarrow{(\mathsf{itr}_2)^*} \cdots \xrightarrow{(\mathsf{itr}_r)^*} (id_n)^* = id_n$$

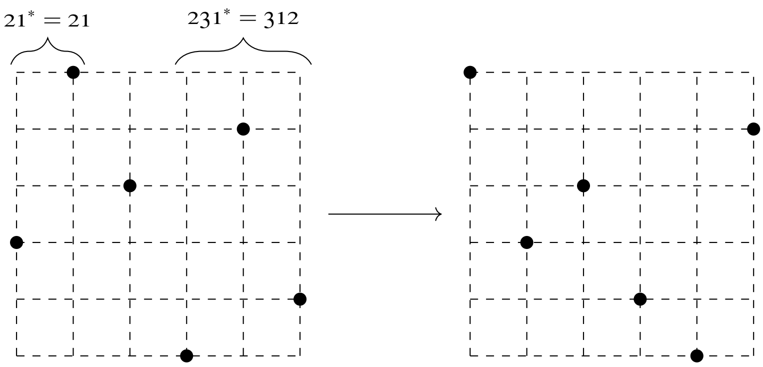
Below is a concrete example of a single step for $\pi = 526314$ and an iteration itr consisting of two cuts associated with the permutations 231 and 21 applied over the segments $\pi_1\pi_2\pi_3 = 526$ and $\pi_5\pi_6 = 14$, respectively. The corresponding iteration itr^* over $\pi^* = 364152$ consists of the two cuts associated with the permutations $312 = 231^*$ and $21 = 21^*$ applied over the segments $\pi^*_{7-3}\pi^*_{7-2}\pi^*_{7-1} = \pi^*_4\pi^*_5\pi^*_6 = 152$ and $\pi^*_{7-6}\pi^*_{7-5} = \pi^*_1\pi^*_2 = 36$.

$$\pi = 526314 \xrightarrow{\mathsf{itr}} 265341 = \pi'$$

$$\pi^* = 364152 \xrightarrow{\mathsf{itr}^*} 634215 = (\pi^*)' = (\pi')^*$$



(a) The diagrams of $\pi = 526314$, $\pi' = 265341$ and the action of itr.



(b) The diagrams of $\pi^* = 364152$, $(\pi^*)' = 634215$ and the action of itr^*.

Figure 14: Example appearing in the proof of Theorem 2.19. Rotate the permutation diagrams on subfigure (a) at $180°$ to obtain those on subfigure (b).

[Proof that $(\pi')^* = (\pi^*)'$]

The diagram of π^* is obtained from the diagram of π by rotating at $180°$. In addition, the iteration itr^* applies the same cuts as the iteration itr, but over the rotated diagram of π^* (see Figure 14). Therefore, the diagrams of $(\pi^*)'$ and π' differ by a $180°$ rotation. Thus, if we rotate the diagram of π' by $180°$, we will get the same diagrams, i.e., $(\pi')^* = (\pi^*)'$.

□

Note that Theorem 2.19 holds not only for cuts, but for any shuffling method Σ that is closed under the * operation.

2.5 Sorting by pop shuffle queues

One can easily see that every shuffle queue of type (ii) (the pop shuffle queues) can always sort at least as many permutations as the shuffle queue of type (i) for the same shuffling method. For instance, we saw, at the beginning of Section 2.3, that $p_n(\mathbb{Q}'_{\text{cuts}}) = O(d^n)$, where $d = 2.32$. It turns out that with the pop shuffle queue for cuts, one can sort all $n!$ permutations in S_n. Below, we prove a more general statement.

Theorem 2.20. If Σ is a shuffling method such that $(\Pi^k_\Sigma)^{-1}$ contains at least one permutation ending in j, for every $j \in [k-1]$ and every $k \geq 2$, then

$$S_n(\mathbb{Q}^{\mathsf{pop}}_\Sigma) = S_n,$$

for every $n \geq 2$. In addition, $\mathbb{Q}^{\mathsf{pop}}_\Sigma$ can sort every permutation using a single pop operation.

Proof. We will use induction on n, relying on a neat observation allowing us to make the induction step. Note that $S_2(\mathbb{Q}_{\Sigma}^{\mathsf{pop}}) = S_2$, since $(\Pi_{\Sigma}^2)^{-1}$ must contain 21 and the permutation 12 trivially belongs to $S_2(\mathbb{Q}_{\Sigma}^{\mathsf{pop}})$. Assume that $n > 2$ and that the statement is true for all $n' < n$. Take an arbitrary permutation π with last element $x \in [n]$ and prefix π'; i.e., $\pi = \pi' x \in S_n$. If $x = n$, then by the induction hypothesis, one can sort π' and then simply push and pop n to sort π. If $x \neq n$, then we know that there exists $\sigma \in (\Pi_{\Sigma}^n)^{-1}$ ending with x. Take one such σ and let $\sigma := \sigma' x$. If we can get output σ' on input π' using $\mathbb{Q}_{\Sigma}^{\mathsf{pop}}$ and only one pop operation, then π would also be sortable by $\mathbb{Q}_{\Sigma}^{\mathsf{pop}}$ and only one pop operation since one can get σ' in the device, push x, and shuffle by applying σ^{-1}.

However, the induction hypothesis gives us that for any input of size $n - 1$, we can always get the identity as an output. In order to get output σ' on input π', we can relabel the elements of π' with $1, \ldots, n$ by looking at σ' as the identity. Formally, since both π' and σ' are permutations of $[n] \setminus \{x\}$, let $\tau \in S_{n-1}$ be the permutation satisfying $\tau\pi' = \sigma'$. By the induction hypothesis, τ can be sorted by $\mathbb{Q}_{\Sigma}^{\mathsf{pop}}$ using only one pop operation. Let itr be one such iteration that sorts τ with a single pop operation. Observe that if we apply the same sequence of operations and permutations as in itr to input π', we will get σ'.

Example 2.21. Shuffling by cuts is a shuffling method having the property described in Theorem 2.20. Let $\pi = 25143$. The last element of π is $3 \in [4]$. Thus, $\exists \sigma \in \Pi_{\text{cuts}}^{-1}$ that ends with 3. Indeed, $\sigma = 45123 \in \Pi_{\text{cuts}}^{-1}$. We have $\pi' = 2514$, $\sigma' = 4512$. The solution to $\tau\pi' = \sigma'$ is $\tau = 4231$. Below is an iteration of $\mathbb{Q}_{\text{cuts}}^{\mathsf{pop}}$ that sorts τ.

$$\begin{pmatrix} 4231 \\ \\ \varepsilon \\ \\ \varepsilon \end{pmatrix} \xrightarrow{\text{push}} \begin{pmatrix} 231 \\ \\ 4 \\ \\ \varepsilon \end{pmatrix} \xrightarrow{\text{push}} \begin{pmatrix} 31 \\ \\ 42 \\ \\ \varepsilon \end{pmatrix} \xrightarrow{\text{push}} \begin{pmatrix} 1 \\ \\ 423 \\ \\ \varepsilon \end{pmatrix} \xrightarrow[\,]{\substack{\text{shuffle} \\ (cut)}} \begin{pmatrix} 1 \\ \\ 234 \\ \\ \varepsilon \end{pmatrix}$$

$$\xrightarrow{\text{push}} \begin{pmatrix} \varepsilon \\ \\ 2341 \\ \\ \varepsilon \end{pmatrix} \xrightarrow{\substack{\text{shuffle} \\ (cut)}} \begin{pmatrix} \varepsilon \\ \\ 1234 \\ \\ \varepsilon \end{pmatrix} \xrightarrow{\substack{\text{pop} \\ (unload)}} \begin{pmatrix} \varepsilon \\ \\ \varepsilon \\ \\ 1234 \end{pmatrix}$$

The same sequence of operations and permutations applied on each shuffle will give an output $\sigma' = 4512$ on input $\pi' = 2514$:

$$\begin{pmatrix} 2514 \\ \\ \varepsilon \\ \\ \varepsilon \end{pmatrix} \xrightarrow{\text{push}} \begin{pmatrix} 514 \\ \\ 2 \\ \\ \varepsilon \end{pmatrix} \xrightarrow{\text{push}} \begin{pmatrix} 14 \\ \\ 25 \\ \\ \varepsilon \end{pmatrix} \xrightarrow{\text{push}} \begin{pmatrix} 4 \\ \\ 251 \\ \\ \varepsilon \end{pmatrix} \xrightarrow{\substack{\text{shuffle} \\ (cut)}} \begin{pmatrix} 4 \\ \\ 512 \\ \\ \varepsilon \end{pmatrix}$$

$$\xrightarrow{\text{push}} \begin{pmatrix} \varepsilon \\ \\ 5124 \\ \\ \varepsilon \end{pmatrix} \xrightarrow{\substack{\text{shuffle} \\ (cut)}} \begin{pmatrix} \varepsilon \\ \\ 4512 \\ \\ \varepsilon \end{pmatrix} \xrightarrow{\substack{\text{pop} \\ (unload)}} \begin{pmatrix} \varepsilon \\ \\ \varepsilon \\ \\ 4512 \end{pmatrix}$$

$\square$

Since the shuffling by $cuts$ satisfies the condition described in Theorem 2.20, we get the following corollary.

Corollary 2.22.

$$p_n(\mathbb{Q}^{\mathsf{pop}}_{\mathrm{cuts}}) = n!.$$

2.5.1 Pop shuffle queues for back-front shuffling methods

In this subsection, we prove Theorem 2.26, which is an analogue of Theorem 2.12 for pop shuffle queues. However, Theorem 2.26 holds for a smaller set of shuffling methods compared to Theorem 2.12, which requires the corresponding shuffling method to have a permutation family consisting of irreducible permutations. Some examples show that if we consider the same collection of shuffling methods for pop shuffle queues, we would not have a similar one-to-one correspondence as in the proof of Theorem 2.12. Nevertheless, we have such a correspondence if we constrain ourselves to shuffling methods having a stronger property which we call the *back-front* property.

Definition 2.23 (Back-front shuffling method)**.** A shuffling method Σ is *back-front* if for every $n \geq 2$, $|\Pi^n_\Sigma| = 1$, i.e., $\Pi^n_\Sigma = \{\sigma_n\}$ for some $\sigma_n \in S_n$ and σ_n begins with n, i.e., the card at the back always goes at the front.

One shuffling method having this property is the rev method defined in Section 2.2. Another example is the shuffling method *top-bottom* defined below. This method simply switches the top and bottom card.

Definition 2.24. The shuffling method *top-bottom*:

$$\forall n \geq 2 : \Pi^n_{\textit{top-bottom}} = \{n23\cdots(n-1)1\}.$$

Example 2.25. Consider the following iteration of $\mathbb{Q}^{\mathsf{pop}}_{top\text{-}bottom}$ over 32415.

$$\begin{pmatrix} 32415 \\ \varepsilon \\ \varepsilon \end{pmatrix} \xrightarrow{\text{push}} \begin{pmatrix} 2415 \\ 3 \\ \varepsilon \end{pmatrix} \xrightarrow{\text{push}} \begin{pmatrix} 415 \\ 32 \\ \varepsilon \end{pmatrix} \xrightarrow{\text{push}} \begin{pmatrix} 15 \\ 324 \\ \varepsilon \end{pmatrix} \xrightarrow{\text{shuffle}} \begin{pmatrix} 15 \\ 423 \\ \varepsilon \end{pmatrix} \xrightarrow{\text{push}} \begin{pmatrix} 5 \\ 4231 \\ \varepsilon \end{pmatrix}$$

$$\xrightarrow{\text{shuffle}} \begin{pmatrix} 5 \\ 1234 \\ \varepsilon \end{pmatrix} \xrightarrow[]{\substack{\text{pop} \\ (unload)}} \begin{pmatrix} 5 \\ \varepsilon \\ 1234 \end{pmatrix} \xrightarrow{\text{push}} \begin{pmatrix} \varepsilon \\ 5 \\ 1234 \end{pmatrix} \xrightarrow[]{\substack{\text{pop} \\ (unload)}} \begin{pmatrix} \varepsilon \\ \varepsilon \\ 12345 \end{pmatrix}$$

Theorem 2.26. For every back-front shuffling method Σ and every $n \geq 2$,

$$p_n(\mathbb{Q}^{\mathsf{pop}}_{\Sigma}) = F_{2n-1},$$

where F_i is the i-th Fibonacci number with $F(1) = F(2) = 1$.

Proof. First, recall that we do not allow two consecutive shuffle operations when using shuffle queues. Therefore, the output after an iteration of $\mathbb{Q}^{\mathsf{pop}}_{\Sigma}$ over a permutation π is determined by the list of segments of π that were shuffled, since $|\Pi^m_{\Sigma}| = 1$ for every $m \geq 2$. For instance, the list of shuffled segments for the iteration of $\mathbb{Q}^{\mathsf{pop}}_{top\text{-}bottom}$ over 324165 shown in Example 2.25 is $([1,3],[1,4],[5,6])$, since exactly three shuffle operations were performed and the device contained the corresponding segment of π before each of them, respectively. A list of shuffled segments l will always be in lexicographical order, i.e.,

$l = ([a_1, b_1], \ldots, [a_r, b_r])$, where $a_i < b_i$, for any $i \in [r]$, $a_i \leq a_j$ whenever $i < j$ and $b_u < b_v$ whenever $a_u = a_v$ and $u < v$. Since we are using pop shuffle queues, if two segments overlap, they must have the same beginning. Thus when describing a list of segments with the same beginning, we will use the shorthand $[a; b_1, b_2, \ldots, b_v]$ to denote $[a, b_1], [a, b_2], \ldots, [a, b_v]$ and we will call such a list a *cluster*.

We are interested in the possible lists of shuffled segments when sorting a permutation with $\mathbb{Q}_{\Sigma}^{\mathsf{pop}}$ or equivalently in the possible lists of clusters. Denote this set of possible lists of clusters for input of size n by LC_n. Note that the set LC_n does not depend on the shuffling method.

The idea of this proof is to show that for any $\pi \in S_n(\mathbb{Q}_{\Sigma}^{\mathsf{pop}})$, there exists a single list of clusters in LC_n, such that any iteration over $\mathbb{Q}_{\Sigma}^{\mathsf{pop}}$ corresponding to it sorts π and vice versa: for any given list of clusters in LC_n, there exists a single $\pi \in S_n(\mathbb{Q}_{\Sigma}^{\mathsf{pop}})$ that can be sorted by the iterations corresponding to this list; i.e., by shuffling the given clusters. This will establish a one-to-one correspondence between the sets $S_n(\mathbb{Q}_{\Sigma}^{\mathsf{pop}})$ and LC_n. Then, we will show that $|LC_n| = F_{2n-1}$.

$[LC_n \to S_n(\mathbb{Q}_{\Sigma}^{\mathsf{pop}})]$ Let $\Pi_{\Sigma}^{m} = \{\sigma_m\}$ and let $l \in LC_n$. Assume that $l = ([a_1, b_1], \ldots, [a_r, b_r])$. Take id_n and apply consecutively $\sigma_{b_r - a_r}^{-1}$ over the segment $[a_r, b_r]$, $\sigma_{b_{r-1} - a_{r-1}}^{-1}$ over the segment $[a_{r-1}, b_{r-1}]$ and so on. After applying $\sigma_{b_1 - a_1}^{-1}$ over $[a_1, b_1]$, we will obtain a permutation $\pi \in S_n$ which can obviously be sorted by $\mathbb{Q}_{\Sigma}^{\mathsf{pop}}$ by shuffling the segments of π in the list l. Therefore, for every $l \in LC_n$, we have only one $\pi \in S_n(\mathbb{Q}_{\Sigma}^{\mathsf{pop}})$ that can be sorted with the iterations corresponding to l.

$[S_n(\mathbb{Q}_{\Sigma}^{\mathsf{pop}}) \to LC_n]$ Let $\pi \in S_n(\mathbb{Q}_{\Sigma}^{\mathsf{pop}})$ and let us assume that π can be sorted by two different iterations it_1 and it_2 over $\mathbb{Q}_{\Sigma}^{\mathsf{pop}}$ corresponding to two different lists of clusters in LC_n, denoted by l_1 and l_2 with their last clusters denoted by $[a_1; b_{11}, \ldots, b_{1c_1}]$ and $[a'_1; b'_{11}, \ldots, b'_{1d_1}]$, respectively. If these last clusters are the same, then before applying the shuffles in each of them, we must have the same permutation in the content of the device. Therefore, without loss of generality, we can assume that $[a_1; b_{11}, \ldots, b_{1c_1}] \neq [a'_1; b'_{11}, \ldots, b'_{1d_1}]$ and that $[a_1, b_{1c_1}] \neq [a'_1, b'_{1d_1}]$. If $b_{1c_1} \neq b'_{1d_1}$, then let $b_{1c_1} < b'_{1d_1}$, without loss of generality. In this case, we will have that the element $\pi_{b'_{1d_1}}$ is not moved anywhere when sorting π by it_1 (and thus $\pi_{b'_{1d_1}} = b'_{1d_1}$). However, when sorting π by it_2, $\pi_{b'_{1d_1}}$ is moved at position a'_1 since the method Σ is back-front and this is the last time this element is moved. Note also that $a'_1 \neq b'_{1d_1}$. Therefore, it_2 does not sort π, which is a contradiction.

Thus $b_{1c_1} = b'_{1d_1}$ and we must have that $a_1 \neq a'_1$. Let $x := b_{1c_1} = b'_{1d_1}$. Then, π_x goes to positions a_1 and a'_1, when we sort π with it_1 and it_2, respectively. This means that $\pi_x = a_1$ and that $\pi_x = a'_1$, but $a_1 \neq a'_1$. This is a contradiction, which shows that any $\pi \in S_n(\mathbb{Q}_\Sigma^{\mathsf{pop}})$ can be sorted by iterations over $\mathbb{Q}_\Sigma^{\mathsf{pop}}$ corresponding to exactly one list of clusters in LC_n.

[Finding $|LC_n|$] The desired correspondence between $S_n(\mathbb{Q}_\Sigma^{\mathsf{pop}})$ and LC_n was established. Therefore, it suffices to get the number of different possible lists of clusters, $|LC_n|$, in order to find $p_n(\mathbb{Q}_\Sigma^{\mathsf{pop}})$. If $l = ([a_1; b_{11}, \ldots, b_{1c_1}], \ldots, [a_m; b_{m1}, \ldots, b_{mc_m}]) \in LC_n$, then l is determined by the $c'_1 + \cdots + c'_m := k$ numbers in $[n]$ comprising l, where we have $c'_j := c_j + 1$ numbers in the j-th cluster and $c'_j \geq 2$ for each $j \in [m]$. We have $a_1 < b_{11} < \cdots < b_{1c_1} < a_2 < \cdots < a_m < b_{m1} < \cdots < b_{mc_m}$ and thus these k numbers can be chosen in $\binom{n}{k}$ ways, where $k \in [2, n]$. The number of compositions $c'_1 + \cdots + c'_m = k$, where each $c'_j \geq 2$ is F_{k-1}, as proved in Lemma 2.27 following this proof. When $k = 0$, we have the empty set of clusters. Therefore, we obtain $|LC_n| = 1 + \sum_{k=2}^{n} \binom{n}{k} F_{k-1}$, which is shown to be equal to F_{2n-1} in Lemma 2.28 via a nice combinatorial argument. $\square$

Lemma 2.27. ([134, Exercise 1.35b]) The number of compositions

$$\alpha_1 + \alpha_2 + \cdots + \alpha_m = k,$$

of an integer $k \geq 2$, where each part $\alpha_j \geq 2$, is given by the Fibonacci number F_{k-1}, where $F_1 = F_2 = 1$.

Lemma 2.28. ([18, Chapter 1, Identity 20])

$$F_{2n-1} = 1 + \sum_{k=2}^{n} \binom{n}{k} F_{k-1}, \tag{2.3}$$

where F_j denotes the j-th Fibonacci number and $F_1 = F_2 = 1$.

2.5.2 A conjecture on Wilf-pop-equivalence

Theorem 2.26 enumerates the sortable permutations for a small subset of pop shuffle queues. One can consider sorting by pop shuffle queues for various other shuffling methods common in the literature, such as the *In-shuffles* and *Monge shuffles* defined in Section 1.4.1.

Definition 2.29 (Wilf-pop-equivalent shuffling methods)**.** The shuffling methods Σ_1 and Σ_2 are *Wilf-pop-equivalent*, if for each $n \geq 1$,

$$p_n(\mathbb{Q}_{\Sigma_1}^{\mathsf{pop}}) = p_n(\mathbb{Q}_{\Sigma_2}^{\mathsf{pop}}).$$

In this section, we formulate and investigate the following conjecture related to the pop shuffle queues for the two methods.

Conjecture 2.30. The In-shuffle and the Monge shuffling methods are Wilf-pop-equivalent.

A first step that may help to establish the conjecture is the next theorem, which confirms it if one has to use a single pop operation. Let $S_n^1(\mathbb{Q}_\Sigma^{\mathsf{pop}})$ be the set of permutations of size n sortable by $\mathbb{Q}_\Sigma^{\mathsf{pop}}$ using only one pop operation and let $p_n^1(\mathbb{Q}_\Sigma^{\mathsf{pop}}) := |S_n^1(\mathbb{Q}_\Sigma^{\mathsf{pop}})|$.

Theorem 2.31. For every $n \geq 1$,

$$p_n^1(\mathbb{Q}_{\text{In-sh}}^{\mathsf{pop}}) = p_n^1(\mathbb{Q}_{\text{Monge}}^{\mathsf{pop}}).$$

In addition, for every $n \geq 3$, $p_n^1(\mathbb{Q}_{\text{In-sh}}^{\mathsf{pop}}) = p_n^1(\mathbb{Q}_{\text{Monge}}^{\mathsf{pop}}) = a_{n-2}$, where $a_1 = 2$, $a_2 = 4$ and $a_n = 3a_{n-2}$ for $n \geq 3$ (sequence A068911 in [118]).

We will show separately, with the next two lemmas, that $p_n^1(\mathbb{Q}_{\text{In-sh}}^{\mathsf{pop}})$ and $p_n^1(\mathbb{Q}_{\text{Monge}}^{\mathsf{pop}})$ are equal to a_{n-2}, for all $n > 4$. This will suffice to establish Theorem 2.31, since one can check directly that the statement of the theorem holds for $n \leq 4$.

Lemma 2.32. $p_n^1(\mathbb{Q}_{\text{Monge}}^{\mathsf{pop}}) = a_{n-2}$, for all $n > 4$.

Proof. Let $\Pi_{\text{Monge}}^i = \{\sigma_i\}$, for $i > 1$. We will need to use permutations of the same size. Thus let

$$\tau_i(x) := \begin{cases} \sigma_i(x), & \text{if } x \leq i, \\ x, & \text{if } x > i. \end{cases} \tag{2.4}$$

be a permutation of size n, for $i \in [2, n]$. Recall that sorting $\pi \in S_n$ with an iteration over $\mathbb{Q}_{\text{Monge}}^{\mathsf{pop}}$ having a single pop, corresponds to a cluster $[1; b_1, b_2, \ldots, n]$, where one performs a shuffle after pushing $b_1, b_2, \ldots, n$ elements, respectively. The output will be $\pi\tau_{b_1}\tau_{b_2}\cdots\tau_n = id_n$. In general, the set of the possible iterations with a single pop over $\mathbb{Q}_{\text{Monge}}^{\mathsf{pop}}$ is described by the set of vectors $(\delta_2, \ldots, \delta_n)$, where

$\delta_i = 0$ or 1, for each $i \in [2,n]$ and the set of possible outputs on input π is described by $\pi\tau_2^{\delta_2}\cdots\tau_n^{\delta_n}$. Note that if $2j+1 < n$ and $j \geq 1$, then $\tau_{2j} = \tau_{2j+1}$.

Therefore, if $n = 2k+1$, for a given k, then the possible outputs on input π are $\pi\tau_2^{\delta_2'}\tau_4^{\delta_4'}\cdots\tau_{2k}^{\delta_{2k}'}$, where $\delta_{2i}' = 0, 1$ or 2, for each $i \in [k]$. If $\pi \in S_n^1(\mathbb{Q}_{\text{Monge}}^{\textsf{pop}})$, then π is a solution to $\pi\tau_2^{\delta_2'}\tau_4^{\delta_4'}\cdots\tau_{2k}^{\delta_{2k}'} = id_n$ for some $(\delta_2', \ldots, \delta_{2k}')$. Thus, $p_n^1(\mathbb{Q}_{\text{Monge}}^{\textsf{pop}})$ is given by the number of different products $\tau_2^{\delta_2'}\tau_4^{\delta_4'}\cdots\tau_{2k}^{\delta_{2k}'}$. We will show that $\tau_2^{\delta_2'}\tau_4^{\delta_4'}\cdots\tau_{2k}^{\delta_{2k}'} \neq \tau_2^{\delta_2''}\tau_4^{\delta_4''}\cdots\tau_{2k}^{\delta_{2k}''}$, if $(\delta_2', \ldots, \delta_{2k}') \neq (\delta_2'', \ldots, \delta_{2k}'')$. This means that it suffices to count the number of different vectors $(\delta_2', \ldots, \delta_{2k}')$ which implies that $p_n^1(\mathbb{Q}_{\text{Monge}}^{\textsf{pop}}) = 3p_{n-2}^1(\mathbb{Q}_{\text{Monge}}^{\textsf{pop}})$ since δ_{2k}' has three possible values.

Assume that $\tau_2^{\delta_2'}\tau_4^{\delta_4'}\cdots\tau_{2k}^{\delta_{2k}'} = \tau_2^{\delta_2''}\tau_4^{\delta_4''}\cdots\tau_{2k}^{\delta_{2k}''}$ for some $(\delta_2', \ldots, \delta_{2k}') \neq (\delta_2'', \ldots, \delta_{2k}'')$. We can further assume that $\delta_{2k}' < \delta_{2k}''$. Therefore, $\tau_2^{\delta_2'}\tau_4^{\delta_4'}\cdots\tau_{2k-2}^{\delta_{2k-2}'} = \tau_2^{\delta_2''}\tau_4^{\delta_4''}\cdots\tau_{2k-2}^{\delta_{2k-2}''}\tau_{2k}^{\delta_{2k}''-\delta_{2k}'}$, where $\delta_{2k}'' - \delta_{2k}' \in \{1, 2\}$. However, $\tau_{2k}(1) = 2k$ and $\tau_{2k}^2(1) = 2k-1$, i.e., $\tau_{2k}^{\delta_{2k}''-\delta_{2k}'}$ moves one of the last two elements to the first position, while neither of $\tau_2, \ldots, \tau_{2k-2}$ moves any of these two elements, which is a contradiction. If $n = 2k$, we can proceed in a similar way. We would still have $p_n^1(\mathbb{Q}_{\text{Monge}}^{\textsf{pop}}) = 3p_{n-2}^1(\mathbb{Q}_{\text{Monge}}^{\textsf{pop}})$ since we have three times more possibilities for the vector $(\delta_2, \ldots, \delta_{2k-2}, \delta_{2k})$, where δ_{2k} is 0 or 1 and δ_{2i} is $0, 1$ or 2 for $i \in [2, k-1]$, compared to $(\delta_2, \ldots, \delta_{2k-2})$, where δ_{2k-2} is 0 or 1 and δ_{2i} is $0, 1$ or 2 for $i \in [2, k-2]$.

$\square$

Lemma 2.33. $p_n^1(\mathbb{Q}_{\text{In-sh}}^{\textsf{pop}}) = a_{n-2}$, for all $n > 4$.

Proof. Let $\Pi^i_{\text{In-sh}} = \{\sigma_i\}$, for $i > 1$ and let

$$\tau_i(x) := \begin{cases} \sigma_i(x), & \text{if } x \le i, \\ x, & \text{if } x > i. \end{cases}$$

be a permutation of size n, for $i \in [2, n]$. Again, we have that $\tau_{2j} = \tau_{2j+1}$, if $2j+1 \le n$. The number $p^1_n(\mathbb{Q}^{\mathsf{pop}}_{\text{In-sh}})$ is given by the number of solutions of $\pi\tau_2^{\delta'_2}\tau_4^{\delta'_4}\cdots\tau_{2k}^{\delta'_{2k}} = id_n$ for some $(\delta'_2, \ldots, \delta'_{2k})$, where $\delta'_{2i} = 0, 1$ or 2, for $i \in [k]$. We will show, again, that $\tau_2^{\delta'_2}\tau_4^{\delta'_4}\cdots\tau_{2k}^{\delta'_{2k}} \neq \tau_2^{\delta''_2}\tau_4^{\delta''_4}\cdots\tau_{2k}^{\delta''_{2k}}$, if $(\delta'_2, \ldots, \delta'_{2k}) \neq (\delta''_2, \ldots, \delta''_{2k})$. Assume the opposite. Assume also that $\delta'_{2k} < \delta''_{2k}$, without loss of generality. Therefore, $\tau_2^{\delta'_2}\tau_4^{\delta'_4}\cdots\tau_{2k-2}^{\delta'_{2k-2}} = \tau_2^{\delta''_2}\tau_4^{\delta''_4}\cdots\tau_{2k-2}^{\delta''_{2k-2}}\tau_{2k}^{\delta''_{2k}-\delta'_{2k}}$, where $\delta''_{2k} - \delta'_{2k} > 0$. The possible values of $\delta''_{2k} - \delta'_{2k}$ are 1 and 2. Now, it suffices to see that $\tau_{2k}(2k-1) = 2k$ and $\tau^2_{2k}(2k-3) = 2k$, while neither of $\tau_2, \ldots, \tau_{2k-2}$ moves the element $2k$. This is a contradiction, implying that $p^1_n(\mathbb{Q}^{\mathsf{pop}}_{\text{In-sh}})$ is equal to the number of different vectors $(\delta'_2, \ldots, \delta'_{2k})$. Thus $p^1_n(\mathbb{Q}^{\mathsf{pop}}_{\text{In-sh}}) = 3p^1_{n-2}(\mathbb{Q}^{\mathsf{pop}}_{\text{In-sh}})$ for both odd and even values of n, in the same way as for Monge shuffles. □

With the next two facts, we give recurrence relations for the number of permutations in $S_n(\mathbb{Q}^{\mathsf{pop}}_{\text{Monge}})$ that end with n and that do not end with n. We also show that we have similar inequalities for these two subsets of $S_n(\mathbb{Q}^{\mathsf{pop}}_{\text{In-sh}})$. Let $p'_n(\mathbb{Q}^{\mathsf{pop}}_{\Sigma}) = |\{\pi \in S_n(\mathbb{Q}^{\mathsf{pop}}_{\Sigma}) \mid \pi_n = n\}|$ and let $p''_n(\mathbb{Q}^{\mathsf{pop}}_{\Sigma}) = |\{\pi \in S_n(\mathbb{Q}^{\mathsf{pop}}_{\Sigma}) \mid \pi_n \neq n\}|$, where Σ is a shuffling method. Denote the number of elements in the device $\mathbb{D}$ before the last pop operation, for an iteration itr over $\mathbb{D}$, by $\text{lps}(\mathsf{itr})$, which stands for *last pop size*. One observation we use is that if π can be sorted by an iteration itr over either $\mathbb{Q}^{\mathsf{pop}}_{\text{Monge}}$ or $\mathbb{Q}^{\mathsf{pop}}_{\text{In-sh}}$, then the last element of π determines whether $\text{lps}(\mathsf{itr})$ is odd or even.

Theorem 2.34. For every $n \geq 1$,

$$p'_n(\mathbb{Q}^{\mathsf{pop}}_{\text{Monge}}) = p_{n-1}(\mathbb{Q}^{\mathsf{pop}}_{\text{Monge}}) + \frac{1}{3}\sum_{j=2}^{\lfloor \frac{n-1}{2} \rfloor} p^1_{2j+1}(\mathbb{Q}^{\mathsf{pop}}_{\text{Monge}}) p_{n-(2j+1)}(\mathbb{Q}^{\mathsf{pop}}_{\text{Monge}}) \tag{2.5}$$

and

$$p'_n(\mathbb{Q}^{\mathsf{pop}}_{\text{In-sh}}) \leq p_{n-1}(\mathbb{Q}^{\mathsf{pop}}_{\text{In-sh}}) + \frac{1}{3}\sum_{j=2}^{\lfloor \frac{n-1}{2} \rfloor} p^1_{2j+1}(\mathbb{Q}^{\mathsf{pop}}_{\text{In-sh}}) p_{n-(2j+1)}(\mathbb{Q}^{\mathsf{pop}}_{\text{In-sh}}). \tag{2.6}$$

Proof. Let $\pi \in S_n(\mathbb{Q}^{\mathsf{pop}}_{\text{Monge}})$, where $\pi_n = n$ and let itr be an iteration sorting π by the given device. Let $\text{lps}(\mathsf{itr}) = k$. The sequence of operations for itr ends either with push, pop or with push, $\mathsf{shuffle}$, pop.

In the first case, the possible prefixes $\pi' = \pi_1 \cdots \pi_{n-1}$ are exactly the permutations in $S_{n-1}(\mathbb{Q}^{\mathsf{pop}}_{\text{Monge}})$, since the iteration itr sorts π' and conversely if we have an iteration itr' that sorts some $\pi' \in S_{n-1}(\mathbb{Q}^{\mathsf{pop}}_{\text{Monge}})$, then $\pi' n$ would be sorted by applying itr' and then adding the operations push, pop at the end. Therefore, we have $p_{n-1}(\mathbb{Q}^{\mathsf{pop}}_{\text{Monge}})$ such permutations π.

In the second case, the last shuffle must leave the element $\pi_n = n$ at the same position. The latter means that $\sigma_k(k) = k$, where $\Pi^k_{\text{Monge}} = \{\sigma_k\}$, which is true if and only if k is odd. Let $k > 1$ be a fixed odd number. Let us also have $\pi' := \text{red}(\pi_{n-k+1} \cdots \pi_n)$. As in the proof of Theorem 2.31, we must have that π' is a solution of the equation $\pi' \tau_2^{\delta_2} \cdots \tau_{k-1}^{\delta_{k-1}} \tau_k^{\delta_k} = id_k$, for a binary vector $(\delta_2, \ldots, \delta_k)$ and where the permutations τ_j are defined by Equation (2.4) in the same proof. Recall that $\delta_j = 1$ if and only if the iteration itr has a shuffle operation immediately after the j-th element of π' is pushed. Take one such solution π' corresponding to the vector $(\delta_2, \ldots, \delta_k)$. We have a shuffle before the last pop, which means that $\delta_k = 1$. If $\delta_{k-1} = 0$, then we must also have $\pi' \tau_2^{\delta_2} \cdots \tau_{k-2}^{\delta_{k-2}} \tau_{k-1} = id_k$ since k is odd and $\tau_{k-1} = \tau_k$. This means that π' can be sorted with an iteration ending with the operations push and pop and thus the same holds for π. These permutations π were already counted in the first case. Therefore, we must have π' for which $\delta_{k-1} = \delta_k = 1$ in order for π to not be counted yet. The number of these permutations π' is the same as the number of different products $\tau_2^{\delta_2} \cdots \tau_{k-2}^{\delta_{k-2}}$, which is $p^1_{k-2}(\mathbb{Q}^{\mathsf{pop}}_{\text{Monge}}) = \frac{1}{3}p^1_k(\mathbb{Q}^{\mathsf{pop}}_{\text{Monge}})$. The permutation $\pi_1 \cdots \pi_{n-k}$ could be any of the permutations in $S_{n-k}(\mathbb{Q}^{\mathsf{pop}}_{\text{Monge}})$. Therefore, summing over all odd values of $k = 2j+1$, we get an inequality similar to Inequality (2.5):

$$p'_n(\mathbb{Q}^{\mathsf{pop}}_{\text{Monge}}) \leq p_{n-1}(\mathbb{Q}^{\mathsf{pop}}_{\text{Monge}}) + \frac{1}{3}\sum_{j=2}^{\lfloor \frac{n-1}{2} \rfloor} p^1_{2j+1}(\mathbb{Q}^{\mathsf{pop}}_{\text{Monge}}) p_{n-(2j+1)}(\mathbb{Q}^{\mathsf{pop}}_{\text{Monge}}). \tag{2.7}$$

All of the steps of the proof so far are applicable to In-shuffles as well. Therefore, we have obtained Inequality (2.6).

It remains to show that instead of Inequality (2.7), one can write an equality. This is true because of the following observation. Assume that π can be sorted by $\mathbb{Q}^{\mathsf{pop}}_{\text{Monge}}$ using two different iterations itr and $\mathsf{itr'}$ with $\text{lps}(\mathsf{itr}) = 2j+1$ and $\text{lps}(\mathsf{itr'}) = 2j'+1$, where $j \neq j'$. Assume also, that $\pi_n = n$ and that $\pi_1 \cdots \pi_{n-1} \notin S_{n-1}(\mathbb{Q}^{\mathsf{pop}}_{\text{Monge}})$. Then, both itr and $\mathsf{itr'}$ must have two shuffle operations after pushing the elements π_{n-1} and π_n, respectively. In addition, $\tau^2_{2v}(1) = 2v - 1$ for each $v \geq 1$. Therefore, since itr sorts π, we must have $\pi_{n-1} = n - (2j+1) + 1 = n - 2j$. However, since $\mathsf{itr'}$ sorts π, we must also have $\pi_{n-1} = n - 2j'$, which is a contradiction. □

Theorem 2.35. For every $n \geq 1$,

$$p''_n(\mathbb{Q}^{\mathsf{pop}}_{\text{Monge}}) = \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} p^1_{2j}(\mathbb{Q}^{\mathsf{pop}}_{\text{Monge}}) p_{n-2j}(\mathbb{Q}^{\mathsf{pop}}_{\text{Monge}}) \tag{2.8}$$

and

$$p_n''(\mathbb{Q}_{\text{In-sh}}^{\mathsf{pop}}) \leq \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} p_{2j}^1(\mathbb{Q}_{\text{In-sh}}^{\mathsf{pop}}) p_{n-2j}(\mathbb{Q}_{\text{In-sh}}^{\mathsf{pop}}). \tag{2.9}$$

Equation (2.8) follows from a property of the Monge shuffle, which we formulate below.

Definition 2.36 (Pop-simple shuffling method)**.** The shuffling method Σ is *pop-simple* if there is no permutation $\pi \in S^1(\mathbb{Q}_\Sigma^{\mathsf{pop}})$, not ending with n, such that $\pi = \pi' \oplus \pi''$ for some π' and π'', where $|\pi'| \geq 2$, $|\pi''| \geq 2$ and $\pi'' \in S^1(\mathbb{Q}_\Sigma^{\mathsf{pop}})$.

Intuitively, if a shuffling method Σ is pop-simple and $\sigma \in S_n(\mathbb{Q}_\Sigma^{\mathsf{pop}})$ does not end with n, then $\operatorname{lps}(\mathsf{itr})$ has the same value for every iteration itr of $\mathbb{Q}_\Sigma^{\mathsf{pop}}$ sorting σ.

Lemma 2.37. The Monge shuffling method is pop-simple.

Proof. Suppose that there exists $\pi \in S_n$, not ending by n, such that $\pi = \pi' \oplus \pi''$ for some π' and π'', such that $|\pi'| \geq 2$, $|\pi''| \geq 2$, and each of π and π'' can be sorted by $\mathbb{Q}_\Sigma^{\mathsf{pop}}$ using a single pop operation. Every iteration sorting a permutation by $\mathbb{Q}_{\text{Monge}}^{\mathsf{pop}}$ that uses a single pop operation can be written as a cluster beginning with the element 1. Consider an arbitrary cluster $[1; b_1, b_2, \ldots, b_v]$ representing an iteration that sorts π. Since π does not end with n, then the last shuffle must be after we push the last element, i.e., $b_v = n$. In addition, we must have $\sigma_n \neq n$, where $\Pi_{\text{Monge}}^n = \{\sigma\}$ and $\sigma = \sigma_1 \cdots \sigma_n$. Note that σ either begins with n (when n is even) or ends with n (when n is odd). Therefore, σ must begin with n, which means that $\pi_n = 1$. However, since $\pi = \pi' \oplus \pi''$, we must have that 1 is among the first $|\pi'|$ elements of π. It cannot be the last element of π', since π'' is non-empty, which is a contradiction. □

Proof of Theorem 2.35. Let $\pi \in S_n(\mathbb{Q}_{\text{Monge}}^{\mathsf{pop}})$, where $\pi_n \neq n$ and let itr be an iteration sorting π by $\mathbb{Q}_{\text{Monge}}^{\mathsf{pop}}$. Note that the sequence of operations for itr must end with $\mathsf{push}, \mathsf{shuffle}, \mathsf{pop}$ since π_n must be moved to another position. This is also the reason that if $\text{lps}(\mathsf{itr}) = k$, then k must be even since all permutations of odd size associated with the Monge shuffle fix its last element. The permutation $\pi_1 \cdots \pi_{n-k}$ could be any of the permutations in $S_{n-k}(\mathbb{Q}^{\mathsf{pop}}_{\text{Monge}})$. Therefore, summing over all even values of $k = 2j$, we get

$$p''_n(\mathbb{Q}^{\mathsf{pop}}_{\text{Monge}}) \leq \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} p^1_{2j}(\mathbb{Q}^{\mathsf{pop}}_{\text{Monge}}) p_{n-2j}(\mathbb{Q}^{\mathsf{pop}}_{\text{Monge}}). \tag{2.10}$$

All of the steps of the proof so far are applicable to In-shuffles, as well, and thus Inequality (2.9) can be obtained analogously.

It remains to show that instead of Equation (2.10), one can write Equation (2.8). Assume that π can be sorted by $\mathbb{Q}^{\mathsf{pop}}_{\text{Monge}}$ using two different iterations itr and $\mathsf{itr'}$ with $\text{lps}(\mathsf{itr}) = 2j$ and $\text{lps}(\mathsf{itr'}) = 2j'$, where $j > j'$. Then, both sequences $\gamma = \text{red}(\pi_{n-2j+1} \cdots \pi_n)$ and $\kappa = \text{red}(\pi_{n-2j'+1} \cdots \pi_n)$ must be permutations of $[2j]$ and $[2j']$, respectively, and they must be sortable with a single pop. However, this would imply that the Monge shuffling method is not pop-simple, because $\gamma = \gamma' \oplus \kappa$ for $\gamma' = \text{red}(\pi_{n-2j+1} \cdots \pi_{n-2j'})$, $|\gamma'| \geq 2$, $|\kappa| \geq 2$ and $\gamma, \kappa \in S^1(\mathbb{Q}^{\mathsf{pop}}_{\text{Monge}})$. This contradicts Lemma 2.37. $\square$

If one can replace Inequality (2.6) and Inequality (2.9) with equations, then one can obtain Conjecture 2.30 using induction and Theorem 2.31. Inequality (2.9) can be replaced by an equation if and only if the In-shuffle method is also pop-simple. It is possible to find permutations π' and π'', for which $\pi = \pi' \oplus \pi''$, $|\pi'|, |\pi''| \geq 2$ and $\pi \in S^1(\mathbb{Q}^{\mathsf{pop}}_{\text{In-sh}})$. For instance, if $\pi = 21 \oplus 62481357$, then $\pi \in S^1(\mathbb{Q}^{\mathsf{pop}}_{\text{In-sh}})$. However, in this example, $\pi'' = 62481357 \notin S^1(\mathbb{Q}^{\mathsf{pop}}_{\text{In-sh}})$. We have performed computer simulations using Theorem 2.35 which show that there is no such permutation $\pi \in S_n$ for $n < 20$ and thus we have an equality in Inequality (2.9) for $n < 20$. Similarly, we have checked that Inequality (2.6) is an equality for $n < 20$. In other words, Conjecture 2.30 holds for $n < 20$.

CHAPTER 3

MOMENTS OF PERMUTATION STATISTICS AND CENTRAL LIMIT THEOREMS

This chapter is based on a joint work with Niraj Khare.

3.1 Aggregates of permutation statistics

We are often interested in the expected value $\mathbb{E}(f)$ of the permutation statistic f, for a permutation chosen uniformly at random from S_n. Obviously, we have $\mathbb{E}(f) = M(f,n)/n!$, where $M(f,n)$ is the *aggregate* of f, defined as

$$M(f,n) := \sum_{\sigma \in S_n} f(\sigma).$$

In this section, we show that $M(f,n)$ is a linear combination of factorials with constant coefficients for each permutation statistic in the class described by Definition 1.43 in Section 1.5. This is an analogue of the results in [37] for aggregates of set partition statistics and those in [96] for aggregates of statistics on matchings. To deal with the constraints, when we have vincular and bivinuclar patterns, we use the same technique to compress numbers used in both of these articles.

Theorem 3.1. Let $f_{\underline{P},Q}$ be a simple statistic of degree m associated with the pattern $\underline{P}$ of size k and the valuation polynomial $Q(s,w) = Q_1(s)Q_2(w)$. Assume that $c = |\boldsymbol{C}(\underline{P})|$ and $d = |\boldsymbol{D}(\underline{P})|$. Then

$$M(f_{\underline{P},Q}, n) = R(n)(n-k)! \tag{3.1}$$

where $R(x)$ is a polynomial of degree no more than $m-c-d$. Equivalently for $n \geq k$, $M(f,n)$ can be expressed as a linear combination of shifted factorials with constant coefficients, i.e.,

$$M(f_{\underline{P},Q},n) = \begin{cases} 0 & \text{if } n < k, \text{ and} \\ \sum_{i=0}^{m-c-d} c_i(n-k+i)! & \text{if } n \geq k \end{cases}, \tag{3.2}$$

for some constants $c_i \in \mathbb{Q}$.

Proof. Let $T_{n,k} := \{(t_1, t_2, \cdots, t_k) \in [n]^k \mid 1 \leq t_1 < t_2 < \cdots < t_k \leq n\}$ be the set of increasing vectors of k numbers in $[n]$. For simplicity, fix n and k and let $T := T_{n,k}$. Note that if $s \in_{\underline{P}} \sigma$ for some $\sigma \in S_n$, then $s \in T$. Let us also define $W := \{(w_1, w_2, \ldots, w_k) \in [n]^k \mid \text{for all } i, j \in [k], \text{ if } w_i = w_j, \text{then } i = j\}$. Note that $|T| = \binom{n}{k}$ and $|W| = n(n-1)\cdots(n-k+1)$. We have

$$\begin{aligned} M(f_{\underline{P},Q},n) = \sum_{\sigma \in S_n} f_{\underline{P},Q}(\sigma) &= \sum_{\sigma \in S_n} \sum_{s \in_{\underline{P}} \sigma} Q_1(s) Q_2(\sigma^{-1}(s)) \\ &= \sum_{s \in T} \sum_{\substack{\sigma \in S_n \\ s \in_{\underline{P}} \sigma}} Q_1(s) Q_2(\sigma^{-1}(s)) = \sum_{s \in T} Q_1(s) \sum_{\substack{\sigma \in S_n \\ s \in_{\underline{P}} \sigma}} Q_2(\sigma^{-1}(s)). \end{aligned}$$

For any $s \in T$, let $G(s) := \{\sigma \in S_n \mid s \in_{\underline{P}} \sigma\}$, and for any $w \in W$, let

$$H(w) := \{\sigma \in S_n \mid (t_1, \ldots, t_k) \in_{\underline{P}} \sigma, \text{ where } \sigma^{-1}(t_i) = w_i, \text{for all } i \in [k]\}.$$

That is, $H(w)$ is the set of permutations in S_n having occurrences of $\underline{P}$ at the positions in w. In addition, for any $s \in T$ and $w \in W$, let $Z(s,w)$ be the set of permutations in S_n having occurrences of $\underline{P}$ at the positions in w and using the values in s. Formally,

$$Z(s,w) := \{\sigma \in S_n \mid s = (t_1, \ldots, t_k) \in_{\underline{P}} \sigma,\ \sigma^{-1}(t_i) = w_i \text{ for all } i \in [k]\}.$$

Clearly, for any $s \in T$ and $w, w' \in W$ for which $w \neq w'$, we have $Z(s,w) \cap Z(s,w') = \emptyset$. Also, note that $G(s) = \cup_{w \in W} Z(s,w)$. Hence, we can rewrite the above equations in the following way:

$$\begin{aligned} M(f_{\underline{P},Q}, n) &= \sum_{s \in T} Q_1(s) \sum_{\substack{\sigma \in S_n \\ s \in_{\underline{P}} \sigma}} Q_2(\sigma^{-1}(s)) = \sum_{s \in T} Q_1(s) \sum_{\sigma \in G(s)} Q_2(\sigma^{-1}(s)) \\ &= \sum_{s \in T} Q_1(s) \sum_{\sigma \in \cup_{w \in W} Z(s,w)} Q_2(w) = \sum_{s \in T} Q_1(s) \left(\sum_{w \in W} Q_2(w) \sum_{\sigma \in Z(s,w)} 1 \right). \end{aligned}$$

Consider any fixed vector of values $s \in T$ and a vector of positions $w \in W$. If $Z(s,w) \neq \emptyset$, then $|Z(s,w)| = (n-k)!$ since the remaining $n-k$ values, except those in s, can be arranged in all the possible ways at the remaining $n-k$ positions, which are not in w. Furthermore, if we define $T' := \{s \in T \mid G(s) \neq \emptyset\}$ and $W' := \{w \in W \mid H(w) \neq \emptyset\}$, then observe that the values in any $s \in T'$ can be at the positions determined by any $w \in W'$ and vice versa. In other words, $Z(s,w) \neq \emptyset$, if and only if $s \in T'$ and $w \in W'$. Therefore,

$$\begin{aligned} M(f_{\underline{P},Q}, n) &= \sum_{s\in T} Q_1(s)\left(\sum_{w\in W} Q_2(w) \sum_{\sigma\in Z(s,w)} 1\right) \\ &= (n-k)!\left(\sum_{s\in T'} Q_1(s)\right)\left(\sum_{w\in W'} Q_2(w)\right). \end{aligned}$$

Consider $s \in T$ and $w \in W$, such that $Z(s,w) \neq \emptyset$. Now, we will use the compression technique, which relies on the following observation: Since $|D(\underline{P})| = d$, every subset of $[n-d]$ of $k-d$ different numbers corresponds to a set of values $s \in T'$ and the correspondence is one-to-one. Formally, let us call $i+1$ a *follower*, if $i \in \boldsymbol{D}(\underline{P})$ and a *non-follower*, if $i \notin \boldsymbol{D}(\underline{P})$. If $g(i) \in [k]$ denotes the index of the i-th non-follower, then let $y_i := t_{g(i)} - (g(i) - i)$. Then, the vector $s \in T'$ determines uniquely the vector $(y_1, \ldots, y_{k-d})$ and one can see that $y_u < y_v$, if $u < v$. Indeed, it suffices to show this for $v = u+1$. In this case we have $y_{u+1} = t_{g(u+1)} - (g(u+1)-(u+1)) > t_{g(u+1)-1} - (g(u+1)-(u+1))$, but we must have that $t_{g(u+1)-1} = t_{g(u)} + (g(u+1) - g(u) - 1)$, because all the numbers between $g(u)$ and $g(u+1)$ are followers. Thus, $y_{u+1} > t_{g(u)} + (g(u+1) - g(u) - 1) - (g(u+1) - (u+1)) = t_{g(u)} - (g(u) - u) = y_u$. Conversely, for any $(y_1, \ldots, y_{k-d}) \in T_{n-d,k-d}$, the vector $(t_1, \ldots, t_k)$ is uniquely determined, since $t_j = y_i + j - i$, where j is the index of the i-th non-follower and $t_j = t_{j-1} + 1$, if j is an index of a follower. Thus Q_1 can be viewed as a polynomial in $y_1, \ldots, y_{k-d}$ and n.

We can proceed in the same way for W' and $\boldsymbol{C}(\underline{P})$. The only difference is that the elements of any $w \in W'$ are not necessarily in increasing order. However, the elements of $\overline{w} = (w_{P^{-1}}(1), \ldots, w_{P^{-1}}(k))$ are always in increasing order and the map $w \mapsto \overline{w}$ is a bijection. Thus, using this map, we can get a set

$W'' \subseteq W$, such that there is a bijection between W' and W'' and a bijection between W'' and $T_{n-c,k-c}$ (by the compression technique). Hence there is a bijection between W' and $T_{n-c,k-c}$ and Q_2 can be viewed as a polynomial in $x_1, \ldots, x_{k-c}$ and n, where $(x_1, \ldots, x_{k-c}) \in T_{n-c,k-c}$. Therefore, we have

$$M(f_{\underline{P},Q}, n) = (n-k)! \sum_{(y_1,\ldots,y_{k-d}) \in T_{n-d,k-d}} \tilde{Q}_1(y_1, \ldots, y_{k-d}, n) \sum_{(x_1,\ldots,x_{k-c}) \in T_{n-c,k-c}} \tilde{Q}_2(x_1, \ldots, x_{k-c}, n),$$

for some polynomials $\tilde{Q}_1$ and $\tilde{Q}_2$ of the same degree as Q_1 and Q_2, respectively. The product of the two sums above yields a polynomial in n of degree at most the sum of the following two terms: the maximum possible degree of n in the product $\binom{n-d}{k-d}\tilde{Q}_1$ and the maximum possible degree of n in the product $\binom{n-c}{k-c}\tilde{Q}_2$. Therefore, the degree of the product is at most

$$k - d + deg(Q_1) + k - c + deg(Q_2) = (deg(Q_1) + deg(Q_2) + 2k) - c - d = m - c - d,$$

since $m = d(f_{\underline{P},Q}) = deg(Q_1) + deg(Q_2) + 2k$.

To see Equation (3.2), let $g_i(n)$ be a polynomial in n defined by $g_i(n) = (n-k+i)!/(n-k)!$. Then g_i is of degree i, and hence $\{g_i(n)\}_{i=0}^{\infty}$ forms a basis of $\mathbb{Q}[n]$. It follows that any polynomial of degree i can be written as a linear combination of $g_0(n), \ldots, g_i(n)$. This implies Equation (3.2). $\square$

Next, we consider any general statistic. Recall that a statistic is a $\mathbb{Q}$-linear combination of simple statistics.

Theorem 3.2. For any statistic f of degree m, there is a positive integer $L \leq \frac{m}{2}$, such that for all $n \geq L$,

$$M(f, n) = U(n)(n-L)!, \tag{3.3}$$

where $U(n)$ is a polynomial of degree no more than $m+L$. Equivalently, if $n \geq L$, then

$$M(f,n) = \sum_{-L \leq i \leq m} \alpha_i (n+i)!, \tag{3.4}$$

for some constants $\alpha_i \in \mathbb{Q}$.

Proof. Assume that

$$f = \sum_{i=1}^{t} h_i f_{\underline{P_i},Q_i},$$

with $h_i \in \mathbb{Q}$. Then, by Theorem 3.1,

$$M(f,n) = \sum_{i=1}^{t} h_i M(f_{\underline{P_i},Q_i}, n) = \sum_{i=1}^{t} h_i R_i(n)(n-k_i)!,$$

where k_i is the size of $\underline{P_i}$ and the degree of $R_i(n)$ is no more than $deg(f_{\underline{P_i},Q_i}) - d_i - c_i \leq m$, where $c_i = |\boldsymbol{C}(\underline{P_i})|$ and $d_i = |\boldsymbol{D}(\underline{P_i})|$. Combining the terms with the same $(n-k_i)!$ yields the equation

$$M(f,n) = \sum_{j=0}^{L} U_j(n)(n-j)!,$$

where $U_j(n)$ is a polynomial of degree no more than m, and $L = \max(k_i) \leq \frac{m}{2}$.

As $\frac{(n-L+i)!}{(n-L)!} = (n-L+i)(n-L+(i-1))\cdots(n-L+1)$ is polynomial in n of degree i, we obtain Equation (3.3) for $n \geq L$. In addition, $\frac{(n-L+i)!}{(n-L)!}$ for $0 \leq i \leq L+m$ forms a basis and Equation (3.4) is obtained by expanding $U(n)$ under the basis $\left\{1, \frac{(n-L+1)!}{(n-L)!}, \frac{(n-L+2)!}{(n-L)!}, \cdots, \frac{(n-L+L+m)!}{(n-L)!}\right\}$. $\square$

Theorem 3.2 allows us to obtain a closed-form expression for $M(f, n)$ (and respectively for $\mathbb{E}(f)$), for any statistic f whenever we know the exact values of $M(f, n)$ for a set of $L+m+1$ values of $n \geq L$, where $m = d(f)$. Then, we can take Equation (3.4) and substitute each of these values for n. We get a system of $L+m+1$ linear equations, where the variables are the numbers α_i, for $i \in [-L, m]$. After we solve it, we have a closed-form expression for $M(f, n)$ as a linear combination of shifted factorials, coming from the same Equation (3.4). We used this approach and implemented a computer program, in order to obtain these closed forms for the aggregates of the statistics given as examples in Chapter 1. Some of the results are listed below.

Example 3.3. (formulas for aggregates of statistics)

a) cnt_{1324}.

Recall that the simple statistic $\mathrm{cnt}_{1324} = f_{\underline{P},Q}$, where $\underline{P} = (1324, \emptyset, \emptyset)$ and $Q = 1$. We have

$$M(\mathrm{cnt}_{1324}, n) = \frac{1}{24}n! - \frac{1}{6}(n+1)! + \frac{1}{8}(n+2)! - \frac{1}{36}(n+3)! + \frac{1}{576}(n+4)!.$$

In fact, a simple linearity of expectation argument gives that $M(\mathrm{cnt}_P, n) = \frac{1}{k!}\binom{n}{k}n!$ for the number of occurrences of any classical pattern P of size k. By using that the so-called Lah numbers, $L(k,j) = \binom{k-1}{j-1}\frac{k!}{j!}$, are the coefficients expressing rising factorials in terms of falling factorials, one can show that

$$M(\mathrm{cnt}_P, n) = \frac{1}{k!}\binom{n}{k}n! = \frac{(-1)^k}{k!}n! + \sum_{j=1}^{k-1} \frac{(-1)^{k-j}}{(j!)^2(k-j)!}(n+j)! + \frac{1}{(k!)^2}(n+k)!.$$

Such a general formula can be derived for an arbitrary bivincular pattern.

b) *Descent drop.*

Recall that for the simple statistic drops $= f_{\underline{P},Q}$, $\underline{P} = (21, \{1\}, \emptyset)$ and $Q(s,w) = Q_1(s)Q_2(w)$, where $Q_1(s) = Q_1(t_1,t_2) = t_2 - t_1$ and $Q_2(w) = 1$. We have

$$M(\text{drops}, n) = -\frac{1}{2}(n+1)! + \frac{1}{6}(n+2)!.$$

c) *Sum of pinnacle squares.*

Recall that the statistic pncSqSum is a sum of the two simple statistics corresponding to the patterns $\underline{P} = (132, \{1,2\}, \emptyset)$ and $\underline{P} = (231, \{1,2\}, \emptyset)$, where the valuation polynomials for both statistics are $Q(s,w) = Q_1(s)Q_2(w)$ with $Q_1(s) = Q_1(t_1,t_2,t_3) = t_3^2$ and $Q_2(w) = 1$. We have

$$M(\text{pncSqSum}, n) = (n+1)! - \frac{5}{4}(n+2)! + \frac{1}{5}(n+3)!.$$

3.2 Higher moments of simple statistics

Our next goal is to show that the higher moments of statistics are also statistics, as defined in Chapter 1. In order to investigate the higher moments, we will need to look at ordered tuples of occurrences of a given pattern. To do that, we will first define a *merge* of patterns, as done originally in [37] for set partitions. In the definition given below, $g(S) := \{g(x) \mid x \in S\}$, where g is a function and S is a set. Also, recall that $A(\pi)$ is the set of distinct pairs of integers (u,v), such that u occurs before v in π.

Definition 3.4 (Merge of patterns)**.** Given are three patterns

$$\underline{P_1} = (x, \boldsymbol{C}(\underline{P_1}), \boldsymbol{D}(\underline{P_1})),\ \underline{P_2} = (y, \boldsymbol{C}(\underline{P_2}), \boldsymbol{D}(\underline{P_2})), \text{ and } \underline{P_3} = (z, \boldsymbol{C}(\underline{P_3}), \boldsymbol{D}(\underline{P_3})),$$

of sizes k_1, k_2 and k_3, respectively. A *merge* of $\underline{P_1}$ and $\underline{P_2}$ onto $\underline{P_3}$ is a pair of increasing functions $m_1 : [k_1] \to [k_3]$ and $m_2 : [k_2] \to [k_3]$, such that

1. $m_1([k_1]) \cup m_2([k_2]) = [k_3]$.
2. for every $i, j \in [k_1]$, $(m_1(i), m_1(j)) \in A(z)$ if and only if $(i, j) \in A(x)$ and for every $i, j \in [k_2]$, $(m_2(i), m_2(j)) \in A(z)$ if and only if $(i, j) \in A(y)$.
3. for every $j \in \boldsymbol{C}(\underline{P_1})$, $z^{-1}(m_1(x_{j+1})) = z^{-1}(m_1(x_j)) + 1$ and for every $j \in \boldsymbol{C}(\underline{P_2})$, $z^{-1}(m_2(y_{j+1})) = z^{-1}(m_2(y_j)) + 1$. In addition,
$$\boldsymbol{C}(\underline{P_3}) = \{z^{-1}(m_1(x_j)) \mid j \in \boldsymbol{C}(\underline{P_1})\} \cup \{z^{-1}(m_2(y_j)) \mid j \in \boldsymbol{C}(\underline{P_2})\}.$$
4. for every $j \in \boldsymbol{D}(\underline{P_1})$, $m_1(j+1) = m_1(j) + 1$ and for every $j \in \boldsymbol{D}(\underline{P_2})$, $m_2(j+1) = m_2(j) + 1$. In addition,
$$\boldsymbol{D}(\underline{P_3}) = \{m_1(j) \mid j \in \boldsymbol{D}(\underline{P_1})\} \cup \{m_2(j) \mid j \in \boldsymbol{D}(\underline{P_2})\}.$$

A merge will be denoted by $m_1, m_2 : \underline{P_1}, \underline{P_2} \to \underline{P_3}$. The *size* of a merge will be the size of the pattern $\underline{P_3}$.

Example 3.5.

Let $\underline{P_1} = (132, \{1\}, \{2\})$, $\underline{P_2} = (21, \emptyset, \emptyset)$ and $\underline{P_3} = (2143, \{2\}, \{3\})$. Define the increasing functions m_1 and m_2 as follows: $m_1(1) = 1$, $m_1(2) = 3$, $m_1(3) = 4$, and $m_2(1) = 1$, $m_1(2) = 2$.

Note that for a merge, the pattern $\underline{P_3}$ may not be uniquely determined by the functions m_1, m_2 and the patterns $\underline{P_1}$, $\underline{P_2}$. For instance, assume that $\underline{P_1} = 321$, $\underline{P_2} = 21$ and $m_1(1) = 1$, $m_1(2) = 2$, $m_1(3) = 4$, $m_2(1) = 3$, $m_1(2) = 4$. Then, $\underline{P_3}$ can be 4321, 4231 or 4213. When we say *merges of two patterns*, $\underline{P_1}$ and $\underline{P_2}$, we will refer to the set of patterns $\underline{P_3}$, for which there exists a merge of $\underline{P_1}$ and $\underline{P_2}$ onto $\underline{P_3}$.

Lemma 3.6. Let $\underline{P_1}$ and $\underline{P_2}$ be two patterns. For any $\sigma \in S_n$, there is a one-to-one correspondence between the following sets.

$$\{(s_1, s_2) : s_1 \in_{\underline{P_1}} \sigma, s_2 \in_{\underline{P_2}} \sigma\} \leftrightarrow \{s_3 \in_{\underline{P_3}} \sigma \mid m_1, m_2 : \underline{P_1}, \underline{P_2} \to \underline{P_3}\}.$$

Proof. Let $\underline{P_1} = (x, C(\underline{P_1}), D(\underline{P_1}))$ and $\underline{P_2} = (y, C(\underline{P_2}), D(\underline{P_2}))$.

($\Longrightarrow$) Assume that $s_1 \in_{\underline{P_1}} \sigma$ and $s_2 \in_{\underline{P_2}} \sigma$. Take the union of the elements of s_1 and s_2 and sort the elements of this union in increasing order. Let s_3 be the resulting increasing vector of numbers in $[n]$. As in the case of matchings and partitions, the maps m_a, for $a = 1, 2$, must be given by the unique function so that $m_a(i) = j$ if and only if the i-th smallest element of s_a equals the j-th smallest element of s_3. If the elements of s_3 form the subsequence $\sigma_{i_1} \cdots \sigma_{i_{k_3}}$ in σ, then let $z = \mathrm{red}(\sigma_{i_1} \cdots \sigma_{i_{k_3}})$ and let $\underline{P_3} = (z, \boldsymbol{C}(\underline{P_3}), \boldsymbol{D}(\underline{P_3}))$, where $\boldsymbol{C}(\underline{P_3}) = \{z^{-1}(m_1(x_j)) \mid j \in \boldsymbol{C}(\underline{P_1})\} \cup \{z^{-1}(m_2(y_j)) \mid j \in \boldsymbol{C}(\underline{P_2})\}$ and $\boldsymbol{D}(\underline{P_3}) = \{m_1(j) \mid j \in \boldsymbol{D}(\underline{P_1})\} \cup \{m_2(j) \mid j \in \boldsymbol{D}(\underline{P_2})\}$.

We will show that $m_1, m_2 : \underline{P_1}, \underline{P_2} \to \underline{P_3}$. One can easily verify that conditions (1) and (2) of Definition 3.4 hold. It remains to show that conditions (3) and (4) of the same definition also hold. We will do this just for $C(\underline{P_1})$ and $D(\underline{P_1})$, since one can proceed in the same way for $C(\underline{P_2})$ and $D(\underline{P_2})$. To check condition (3), it suffices to show that for every $j \in C(\underline{P_1})$, $z^{-1}(m_1(x_{j+1})) = z^{-1}(m_1(x_j)) + 1$.

Indeed, the positions of the elements corresponding to x_j and x_{j+1} in every occurrence of $\underline{P_1}$, must be consecutive. Thus, since $s_1 \in_{\underline{P_1}} \sigma$, the positions of $m_1(x_j)$ and $m_1(x_{j+1})$ in σ, and consequently in z, must be consecutive, because z is the reduction of s_3, which is the union of s_1 and s_2. Also, if $j \in D(\underline{P_1})$, then $t_{j+1} = t_j + 1$, where $s_1 = (t_1, \ldots, t_{k_1})$. Therefore, these two elements have consecutive values in s_3, as well, i.e., $m_1(j+1) = m_1(j) + 1$. With that, we showed that $m_1, m_2 : \underline{P_1}, \underline{P_2} \to \underline{P_3}$. Now, it is easy to check that $s_3 \in_{\underline{P_3}} \sigma$.

($\Longleftarrow$) Let $s_3 \in_{\underline{P_3}} \sigma$, where $s_3 = (t_1, t_2, \ldots, t_{k_3})$ is an increasing vector, $m_1, m_2 : \underline{P_1}, \underline{P_2} \to \underline{P_3}$ and $\underline{P_3} = (z, \boldsymbol{C}(\underline{P_3}), \boldsymbol{D}(\underline{P_3}))$. Define $s_1 := t|_{m_1,k_1}$ and $s_2 := t|_{m_2,k_2}$, where $t|_{h,k} := (t_{h(1)}, t_{h(2)}, \ldots, t_{h(k)})$. We must show that $t|_{m_1,k_1} \in_{\underline{P_1}} \sigma$. One can similarly show that $t|_{m_2,k_2} \in_{\underline{P_2}} \sigma$. Condition (2) of Definition 3.4 implies that the elements of $t|_{m_1,k_1}$ are in the same relative order in σ as the elements of $\underline{P_1}$. Now, assume that $j \in \boldsymbol{C}(\underline{P_1})$. We have to show that the positions of the elements $t_{m_1(x_j)}$ and $t_{m_1(x_{j+1})}$ in σ are consecutive. According to condition (3) of Definition 3.4, we have $z(m_1(x_{j+1})) = z(m_1(x_j)) + 1$, i.e., $m_1(x_j)$ and $m_1(x_{j+1})$ have consecutive positions in z and $z^{-1}(m_1(x_j)) \in \boldsymbol{C}(\underline{P_3})$. Therefore, these positions must be also consecutive in σ since $s_3 \in_{\underline{P_3}} \sigma$. Finally, assume that $j \in \boldsymbol{D}(\underline{P_1})$. We have to show that $t_{m_1(j+1)} = t_{m_1(j)} + 1$. According to condition (4) of Definition 3.4, we must have that $m_1(j) \in \boldsymbol{D}(\underline{P_3})$ and $m_1(j+1) = m_1(j) + 1$. Since $s_3 \in_{\underline{P_3}} \sigma$, we have $t_{m_1(j)} + 1 = t_{m_1(j)+1} = m_1(j+1)$. □

Assume that f is a simple statistic associated with the pattern $\underline{P_1}$ and valuation function $Q_1 Q_1'$, whereas g is a simple statistic associated with the pattern $\underline{P_2}$ and valuation function $Q_2 Q_2'$.

Assume, also, that $m_1, m_2 : \underline{P_1}, \underline{P_2} \to \underline{P_3}$ for some m_1, m_2 and $\underline{P_3}$. If $s_3 = (t_1, t_2, \cdots, t_{k_3}) \in_{\underline{P_3}} \sigma$ and $w_3 = (\sigma^{-1}(t_1), \sigma^{-1}(t_2), \ldots, \sigma^{-1}(t_{k_3}))$, then let us define

$$Q_{m_1,m_2,Q_1,Q_2}(s_3) := Q_1(t|_{m_1,k_1}, n)Q_2(t|_{m_2,k_2}, n).$$

and

$$Q'_{m_1,m_2,Q_1,Q_2}(w_3) := Q'_1(\sigma^{-1}(t|_{m_1,k_1}), n)Q'_2(\sigma^{-1}(t|_{m_2,k_2}), n).$$

Theorem 3.7. Let $\mathbb{S}\mathbb{t}\mathbb{a}\mathbb{t}$ be the set of all permutation statistics thought of as functions $f : \cup_n S_n \to \mathbb{Q}$. Then $\mathbb{S}\mathbb{t}\mathbb{a}\mathbb{t}$ is closed under the operations of point-wise scaling, addition and multiplication. Thus, if f, $g \in \mathbb{S}\mathbb{t}\mathbb{a}\mathbb{t}$ and $a \in \mathbb{Q}$, then there exist permutation statistics h_a, h_+ and h_* so that for all permutations $\sigma \in \mathbb{S}\mathbb{t}\mathbb{a}\mathbb{t}$,

$$\begin{aligned} af(\sigma) &= h_a(\sigma), \\ f(\sigma) + g(\sigma) &= h_+(\sigma), \\ f(\sigma)g(\sigma) &= h_*(\sigma). \end{aligned}$$

Furthermore, we have the following inequalities for the degrees: $d(h_a) \leq d(f), d(h_+) \leq \max\{d(f), d(g)\}$ and $d(h_*) \leq d(f) + d(g)$.

Proof. The addition of two statistics is obviously a statistic by definition and thus h_+ exists. Using this, one can easily see that it suffices to show the existence of h_a and h_*, when f and g are simple statistics.

Here is the argument for the existence of h_a. The existence of h_* follows in a similar way. Assume that $ag(\sigma)$ is a statistic, for every simple statistic g. Then, when $f = \sum_{j=1}^{m} f_j$, for some simple statistics f_j, we have that $h_a = af = a\sum_{j=1}^{m} f_j = \sum_{j=1}^{m} af_j$ is a sum of statistics and therefore is a statistic itself.

If f corresponds to the pattern $\underline{P}$ and the valuation function $Q(s,w) = Q_1(s)Q_2(w)$, then let h_a be the simple statistic corresponding to the same pattern $\underline{P}$ and valuation function $Q'(s,w) = aQ_1(s)Q_2(w) = Q_1'(s)Q_2(w)$. Clearly, h_a is a statistic. To establish the fact that the product of two simple statistics is a statistic, we need Lemma 3.6. Let f and g have associated patterns $\underline{P_1}$, $\underline{P_2}$ and valuations functions Q_1Q_1' and Q_2Q_2', respectively. For any positive integer n, let $\sigma \in S_n$ and consider

$$f_{\underline{P_1},Q_1}(\sigma)g_{\underline{P_2},Q_2}(\sigma) = \sum_{s_1 \in \underline{P_1}\sigma} Q_1(s_1)Q_1'(\sigma^{-1}(s_1)) \sum_{s_2 \in \underline{P_2}\sigma} Q_2(s_2)Q_2'(\sigma^{-1}(s_2))$$

$$\overset{\text{(by Lemma 3.6)}}{=} \sum_{\underline{P_3}} \left(\sum_{s_3 \in \underline{P_3}\sigma} \left(\sum_{m_1,m_2:\underline{P_1},\underline{P_2}\to\underline{P_3}} Q_{m_1,m_2,Q_1,Q_2}(s_3)Q'_{m_1,m_2,Q_1,Q_2}(\sigma^{-1}(s_3)) \right)\right) = \sum_{\underline{P_3}} f_{\underline{P_3},\tilde{Q}},$$

where

$$\tilde{Q}(s_3) = \sum_{m_1,m_2:\underline{P_1},\underline{P_2}\to\underline{P_3}} Q_{m_1,m_2,Q_1,Q_2}(s_3)Q'_{m_1,m_2,Q_1,Q_2}(\sigma^{-1}(s_3))$$

for the fixed $\underline{P_1}, \underline{P_2}$ and $\underline{P_3}$. We get that the product fg is a finite sum of statistics and thus, it is a statistic itself. Indeed, this sum is finite since the number of patterns $\underline{P_3}$ that one can get as a merge of $\underline{P_1}$ and $\underline{P_2}$ is finite. Note that the bounds on the degrees of the statistics h_a, h_+ and h_* follow directly from our proof and the definitions. □

We will also need a generalization of Definition 3.4.

Definition 3.8. Let $\underline{P_1}$,$\underline{P_2}$, …, $\underline{P_l}$ be l patterns, where k is the size of the pattern $\underline{P}$ and for each $i \in [l]$, k_i is the size of the pattern $\underline{P_i}$. If we have the increasing functions $m_1 : [k_1] \longrightarrow [k]$, $m_2 : [k_2] \longrightarrow [k]$, $\ldots$, $m_l : [k_l] \longrightarrow [k]$, then a *merge* of these l patterns corresponding to the listed functions is denoted by $m_1, m_2, \ldots, m_l : \underline{P_1}, \underline{P_2}, \ldots, \underline{P_l} \longrightarrow \underline{P}$ or by the shorthand $\mathcal{M}_l : \Pi_l \longrightarrow \underline{P}$. The pattern $\underline{P}$ in such a merge will be a union of l subsequences, which are order-isomorphic to $\underline{P_1}, \underline{P_2}, \ldots, \underline{P_l}$ and determined by the functions $m_1, m_2, \ldots, m_l$.

Similarly, for any $\sigma \in S_n$ one can establish an analogue of Lemma 3.6. We state this result without a proof.

Lemma 3.9. Assume that we have the r patterns $(\underline{P_1}, \boldsymbol{C}(\underline{P_1}), \boldsymbol{D}(\underline{P_1}))$, $(\underline{P_2}, \boldsymbol{C}(\underline{P_2}), \boldsymbol{D}(\underline{P_2}))$, $\ldots$, $(\underline{P_r}, \boldsymbol{C}(\underline{P_r}), \boldsymbol{D}(\underline{P_r}))$. There is a one-to-one correspondence between the following sets.

$$\{(s_1, s_2, \ldots, s_r) \mid s_1 \in_{\underline{P_1}} \sigma, s_2 \in_{\underline{P_2}} \sigma, \ldots, s_r \in_{\underline{P_r}} \sigma\}$$

$$\leftrightarrow \{s \in_{\underline{P}} \sigma \mid m_1, m_2, \ldots, m_r : \underline{P_1}, \underline{,} \ldots, \underline{P_r} \longrightarrow P\}.$$

Using this lemma, one can obtain analogously that the product of r statistics of degrees $d_1, \ldots, d_r$ is a statistic of degree not more than $\sum_{j=1}^{r} d_j$. We use this observation to obtain the following result.

Theorem 3.10. Let f be any statistic of degree m. Then, for any positive integer r,

$$M(f^r, n) = \sum_{-I \le i \le J} \alpha_i (n+i)!, \tag{3.5}$$

where I and J are constants that satisfy $-I \geq \frac{-rm}{2}$, $J \leq mr$ and $n \geq I$, and the α_i's are rational constants.

Proof. Let $f = \sum_{i=0}^{t} \beta_i f_{\underline{P_i},Q_i}$. We have

$$M(f^r,n) = \sum_{\sigma\in S_n}\left(\sum_{i=1}^{t}\beta_i f_{\underline{P_i},Q_i}(\sigma)\right)^r = \sum_{\sigma\in S_n}\sum_{\underline{P^*}}\gamma_j\left(\sum_{s\in_{\underline{P^*}}\sigma}\left(\sum_{\mathcal{M}_r:\Pi_r\to\underline{P^*}}\prod_{i=1}^{r}Q_i(t\mid_{m_i,k_i},\sigma^{-1}(t\mid_{m_i,k_i}))\right)\right)$$
$$\overset{\text{(by Lemma 3.9)}}{=} \sum_{\sigma\in S_n}\sum_{\underline{P^*}}\gamma_j f_{\underline{P^*},\tilde{Q}}(\sigma) = \sum_{\underline{P^*}}\gamma_j M(f_{\underline{P^*},\tilde{Q}},n), \tag{3.6}$$

for some constants $\gamma_j \in \mathbb{Q}$. Each of the statistics $f_{\underline{P^*},\tilde{Q}}$ is a summation of products of r statistics, with each of them being of degree not more than m. Thus, $f_{\underline{P^*},\tilde{Q}}$ is a statistic of degree not more than rm, for every $\underline{P^*}$. Therefore, by Theorem 3.2, we get

$$M(f^r,n) = \sum_{-L\le i\le rm}\alpha_i(n+i)! \tag{3.7}$$

where $L \le \frac{m}{2}$. □

In order to establish Lemma 3.12, which is an important special case of Theorem 3.10, we will need the lemma below.

Lemma 3.11. Consider a merge of the vincular patterns $\underline{P_1} = (x, \boldsymbol{C}(\underline{P_1}))$ and $\underline{P_2} = (y, \boldsymbol{C}(\underline{P_2}))$ onto $\underline{P_3} = (z, \boldsymbol{C}(\underline{P_3}))$, where x, y and z are of sizes k_1, k_2 and k_3, respectively and the values of $|\boldsymbol{C}(\underline{P_1})|$, $|\boldsymbol{C}(\underline{P_2})|$ and $|\boldsymbol{C}(\underline{P_3})|$ are c_1, c_2 and c_3, respectively. Then,

$$k_3 - c_3 \le (k_1+k_2) - (c_1+c_2).$$

Proof. Part (3) of Definition 3.4 allows us to write the following:

$$k_3 - c_3 = (k_1 + k_2 - |m_1([k_1]) \cap m_2([k_2])|) - \Big(c_1 + c_2 - |\{m_1(x_i) \mid i \in \boldsymbol{C}(\underline{P_1})\} \cap \{m_2(y_j) \mid j \in \boldsymbol{C}(\underline{P_2})\}|\Big) =$$

$$(k_1 + k_2) - (c_1 + c_2) - \Big[|m_1([k_1]) \cap m_2([k_2])| - |\{m_1(x_i) \mid i \in \boldsymbol{C}(\underline{P_1})\} \cap \{m_2(y_j) \mid j \in \boldsymbol{C}(\underline{P_2})\}|\Big].$$

Thus, it suffices to show that

$$|m_1([k_1]) \cap m_2([k_2])| - |\{m_1(x_i) \mid i \in \boldsymbol{C}(\underline{P_1})\} \cap \{m_2(y_j) \mid j \in \boldsymbol{C}(\underline{P_2})\}| \geq 0,$$

but the latter is clearly true since $\boldsymbol{C}(\underline{P_1})$ and $\boldsymbol{C}(\underline{P_2})$ are subsets of $[k_1]$ and $[k_2]$, respectively. □

Theorem 3.12. If $\underline{P}$ is a vincular pattern of size k, such that $|\boldsymbol{C}(\underline{P})| = c$, then

$$M(\mathrm{cnt}^r_{\underline{P}}, n) = \sum_{0 \leq i \leq r(k-c)} \alpha_i (n+i)!, \tag{3.8}$$

for $n \geq rk$.

Proof. One can easily prove the following equality (Lemma 3.15, proved in the next section, gives a generalisation):

$$M(\mathrm{cnt}_{\underline{P}}, n) = \frac{\binom{n-c}{k-c}}{k!} n!.$$

Since $\binom{n-c}{k-c}$ is a polynomial in n of degree $k-c$, the statement of the lemma holds, when $r = 1$. For bigger values of r, we can look at Equation (3.6) and plug in $t = 1, \beta_1 = 1$ and $Q = 1$ for all valuation functions Q, as well as $\underline{P_1} = \underline{P_2} = \cdots = \underline{P_r} = \underline{P}$. We will get that

$$M(\mathrm{cnt}^r_{\underline{P}}, n) = \sum_{\underline{P^*}} \delta_j M(\mathrm{cnt}_{\underline{P^*}}, n), \tag{3.9}$$

where the summation is over all possible merges $\underline{P^*}$ of r copies of $\underline{P}$ and where δ_j are some rational constants. Using Lemma 3.11, we can see that each of the patterns $\underline{P^*} = (z, \boldsymbol{C}(\underline{P^*}))$ is a vincular pattern with $|z| - |\boldsymbol{C}(\underline{P^*})| \leq r(k-c)$. Therefore, each of the aggregates $M(\mathrm{cnt}_{\underline{P^*}}, n)$ can be written in the form, as in the right side of Equation (3.8). After we substitute these forms in the right side of Equation (3.9) and regroup, we see that the claim holds. □

Theorem 3.10 and Theorem 3.12 generalize a result of Zeilberger [145, Main formula]. What he proved is that for any classical pattern $\underline{P}$ of size k, $\mathbb{E}(\mathrm{cnt}^r_{\underline{P}})$ is a polynomial of degree rk. In the same article, he used this observation to get the polynomials for the second and the third moments of the statistic $\mathrm{cnt}_{\underline{P}}$, for various classical patterns $\underline{P}$. To do that, he implemented a computer program that fits the actual values of this statistic for $0, 1, \ldots, rk$ to a polynomial of degree rk. Below, we give explicit expressions for the aggregates when $r = 2$ (and respectively, the second moments) of some of the statistics introduced in Section 1.5. We use the same approach by fitting small values of these statistics to the right side of Theorem 3.10 or Theorem 3.12, in order to find the coefficients α_i.

Example 3.13. (formulas for aggregates of higher moments)

a) Second moment of the double ascents.

$$M(\mathrm{cnt}^2_{\underline{123}}, n) = -\frac{1}{12}n! - \frac{1}{15}(n+1)! + \frac{1}{36}(n+2)!.$$

b) Second moment of $\mathrm{cnt}_{\underline{P^*}}$, where $\underline{P^*} = (312, \{2\}, \{2\})$.

$$M(\mathrm{cnt}^2_{\underline{P^*}}, n) = \frac{1}{2}n! - \frac{9}{28}(n+1)! + \frac{29}{672}(n+2)! + \frac{11}{10080}(n+3)! - \frac{1}{45360}(n+4)!.$$

Several important simple statistics have unit valuation function associated to them, i.e., $Q(s,w) = 1$. For these cases, we give the following important corollary from Theorem 3.10, which is an analogue of [96, Proposition 3.5] and will be substantially used in the next two sections.

Corollary 3.14. Let $\underline{P}$ be a pattern of size k with $|\boldsymbol{C}(\underline{P})| = c$, $|\boldsymbol{D}(\underline{P})| = d$ and unit valuation function. Then,

$$M(\mathrm{cnt}^r_{\underline{P}}, n) = \sum_{\tilde{k},\tilde{c},\tilde{d}} w^{(r)}_{\tilde{k},\tilde{c},\tilde{d}} \binom{n-\tilde{c}}{\tilde{k}-\tilde{c}} \binom{n-\tilde{d}}{\tilde{k}-\tilde{d}} (n-k)!, \tag{3.10}$$

where $w^{(r)}_{\tilde{k},\tilde{c},\tilde{d}}$ is the number of ways to merge r copies of $\underline{P}$ and get a pattern $\underline{P^*}$ of size $\tilde{k}$, with $|C(\underline{P^*})| = \tilde{c}$, $|D(\underline{P^*})| = \tilde{d}$ and where $k \leq \tilde{k} \leq rk$, $c \leq \tilde{c} \leq rc$ and $d \leq \tilde{d} \leq rd$.

Proof. Take Equation (3.6) in the proof of Theorem 3.10 and plug in $t = 1$, $\beta_1 = 1$, $Q = 1$ for all valuation functions Q and $\underline{P_1} = \underline{P_2} = \cdots \underline{P_r} = \underline{P}$. Then, $M(\mathrm{cnt}^r_{\underline{P}}, n) = \sum_{\underline{P^*}} \gamma_j M(f_{\underline{P^*},\tilde{Q}}, n) = \sum_{\underline{P^*}} \gamma'_j M(\mathrm{cnt}_{\underline{P^*}}, n)$, for some rational constants γ'_j. In addition, use Lemma 3.15 to get that

$$M(\mathrm{cnt}_{\underline{P}}, n) = \frac{\binom{n-\tilde{c}}{\tilde{k}-\tilde{c}}\binom{n-\tilde{d}}{\tilde{k}-\tilde{d}}}{n_{(\tilde{k})}} n! = \binom{n-\tilde{c}}{\tilde{k}-\tilde{c}} \binom{n-\tilde{d}}{\tilde{k}-\tilde{d}} (n-\tilde{k})!,$$

for every pattern $\underline{P}$ of size $\tilde{k}$, with $|C(\underline{P})| = \tilde{c}$ and $|D(\underline{P})| = \tilde{d}$. □

3.3 Descents and minimal descents: explicit formulas for the higher moments

The results from the previous section can be used to obtain an explicit formula for the r-th moment of some permutation statistics. In this section, we illustrate how this can be done for the descents and the minimal descents statistics. We will use the following simple lemma.

Lemma 3.15. For any bivincular pattern $\underline{P}$ of size k, such that $|\boldsymbol{C}(\underline{P})| = c$ and $|\boldsymbol{D}(\underline{P})| = d$,

$$\mathbb{E}(\mathrm{cnt}_{\underline{P}}, n) = \frac{\binom{n-c}{k-c}\binom{n-d}{k-d}}{n_{(k)}}.$$

Proof. Let I be the set of possible positions for an occurrence of $\underline{P}$ in a permutation of size n. Similarly, let J be the set of possible values of the numbers in such an occurrence. By linearity of expectation, we have that

$$\mathbb{E}(\mathrm{cnt}_{\underline{P}}, n) = \sum_{i \in I, j \in J} X_{i,j},$$

where the random variable $X_{i,j} := 1$, if the set of possible values with index j are at the set of possible positions with index i, and these values are in the relative order determined by the permutation P. Otherwise, $X_{i,j} := 0$. Note that when we choose a permutation of size n at random, $\mathbb{E}(X_{i,j}) = \frac{1}{n_{(k)}}$. Also, note that $|I| = \binom{n-c}{k-c}$ and $|J| = \binom{n-d}{k-d}$. □

Consider the statistic des $= \mathrm{cnt}_{\underline{21}}$. It is well known that the number of permutations of size n having k descents is given by the *Eulerian numbers* and the corresponding distribution is called *Eulerian distribution*. A comprehensive source dedicated to Eulerian numbers is the book [119]. Its preface and the notes at the end of Chapter 1 provide a good historical overview. A recent article by Hwang et al. gives a complicated recurrence relation as a way to calculate the higher moments of the Eulerian distribution and a family of other distributions with generating functions satisfying a similar relation (see [87, Section 2.2]). Below, we give a direct summation formula for the r-th moment of the Eulerian distribution.

Theorem 3.16. Consider a random permutation of size n and $r \geq 1$. Then,

$$\mathbb{E}(\mathrm{des}^r) = \sum_{m=2}^{\min(n,2r)} \sum_{u=1}^{\lfloor \frac{m}{2} \rfloor} \left(\sum_{w=0}^{m-u} (-1)^w \binom{m-u}{w} (m-u-w)^r \right) \left(\sum_{\substack{q_1+\cdots+q_u=m \\ q_i \geq 2}} \binom{m}{q_1,\ldots,q_u} \right) \frac{\binom{n-(m-u)}{u}}{m!}.$$

Proof. Use Corollary 3.14 and note that for $\underline{P} = \underline{21}$, $d = 0$ and $c = 1$. Let us find the numbers $w^{(r)}_{\tilde{k},\tilde{c}}$ for the pattern $\underline{21}$. We will need to sum over all possible merges $\underline{P^*}$ of r copies of $\underline{P}$, depending on their size $\tilde{k}$ and the value $\tilde{c}$ of $|\boldsymbol{C}(\underline{P^*})|$. Instead of $\tilde{k}$, we will write m. Any of the patterns $\underline{P^*}$ can have between $m = 2$ and $m = 2r$ letters. For a fixed m, any such pattern can be comprised of u segments of letters at consecutive positions, where $1 \leq u \leq \lceil \frac{m}{2} \rceil$. For example, $q = \underline{43}\,\underline{61}\,\underline{752}$ has size $m = 7$ and is comprised of three segments of letters at consecutive positions, namely 43, 61 and 752. Note that getting a pattern $\underline{P}$ with u segments requires merging at least $m-u$ copies of the pattern $\underline{21}$ since a segment of size h requires merging at least $h-1$ copies of $\underline{21}$. For instance, the segment $\underline{752}$ in the pattern q above can be obtained after merging multiple copies of $\underline{21}$, corresponding either to $\underline{75}$ or to $\underline{52}$ and at least one copy corresponding to each of them. In general, for a merge with u segments, each of the r copies of the descent pattern $\underline{21}$ must correspond to one out of $m-u$ pairs of consecutive elements and we must have at least one copy for each of these pairs. The inclusion-exclusion principle gives us

$\sum_{w=0}^{m-u}(-1)^w\binom{m-u}{w}(m-u-w)^r$ ways to achieve that. In addition, every segment must be a decreasing sequence of elements. If the lengths of the segments in the pattern $\underline{P^*}$ are denoted by $q_1,\dots,q_u$, then we must have $q_1+\cdots+q_u=m$ and $q_i\geq 2$ for each $1\leq i\leq u$. Thus, for every such composition of m, we can choose the numbers in each of the segments in $\binom{m}{q_1,\dots,q_u}$ ways. Finally, for every pattern $\underline{P^*}$ with u segments, $|\boldsymbol{C}(\underline{P^*})|=m-u$. Therefore

$$w^{(r)}_{\tilde{k},\tilde{c}}=w^{(r)}_{m,m-u}=\sum_{u=1}^{\lfloor\frac{m}{2}\rfloor}\left(\sum_{w=0}^{m-u}(-1)^w\binom{m-u}{w}(m-u-w)^r\right)\sum_{\substack{q_1+\cdots+q_u=m\\ q_i\geq 2}}\binom{m}{q_1,\dots,q_u}$$

and $M(\mathrm{cnt}_{\underline{P^*}},n)=\frac{\binom{n-(m-u)}{u}}{m!}n!$, by Lemma 3.15. Our goal is to find $\mathbb{E}(\mathrm{des}^r)$, so we are dividing both sides by $n!$ to obtain the desired formula. □

Similarly, we can obtain the moments of the minimal descents statistic $\mathrm{cnt}_{\underline{P}}$, where $\underline{P}=(21,\{1\},\{1\})$. These are descents, such that the two numbers in them are consecutive. In the literature, this statistic is also known as *adjacency* and we will denote it by adj. The following Theorem will be used in Section 3.4.3.

Theorem 3.17. Consider a random permutation of size n and $r\geq 1$. Then,

$$\mathbb{E}(\mathrm{adj}^r)=\sum_{m=2}^{\min(n,2r)}\sum_{u=1}^{\lfloor\frac{m}{2}\rfloor}\left(\left(\sum_{w=0}^{m-u}(-1)^w\binom{m-u}{w}(m-u-w)^r\right)\binom{m-u-1}{u-1}u!\frac{\binom{n-(m-u)}{u}^2}{n_{(m)}}\right).$$

Proof. Proceed as in the proof of Theorem 3.16. One difference is that now, for a pattern $\underline{P^*}$ of size m with u segments, the values of the numbers in each segment must be consecutive. Thus, instead of

$\sum_{\substack{q_1+\cdots+q_u=m\\ q_i\geq 2}} \binom{m}{q_1,\ldots,q_u}$ possible ways to determine the numbers in a pattern with u segments, we have just $u!$ such segments for every solution of $q_1 + \cdots + q_u = m$, where $q_i \geq 2$. By using the stars and bars model, one can see that the number of these solutions is exactly $\binom{m-u-1}{u-1}$. In addition, one can see that $|\boldsymbol{C}(\underline{P}^*)| = |\boldsymbol{D}(\underline{P}^*)| = m - u$, for every pattern $\underline{P}^*$ with u segments and therefore by Lemma 3.15, we get $M(\mathrm{cnt}_{\underline{P}^*}, n) = \frac{\binom{n-(m-u)}{u}^2}{n_{(m)}}$. □

3.4 Central limit theorems for $\mathrm{cnt}_{\underline{P}}$.

In this section, we will reprove some limiting laws for the random variable $\mathrm{cnt}_{\underline{P}}$, which counts the number of occurrences of the pattern $\underline{P}$ in a given permutation.

3.4.1 Classical patterns

Recall that if $\boldsymbol{C}(\underline{P})$ and $\boldsymbol{D}(\underline{P})$ are empty, then $\underline{P}$ is a classical pattern. The limiting normality of $\mathrm{cnt}_{\underline{P}}$, when $\underline{P}$ is a classical pattern was first established by Bóna [24]. He uses the method of *dependency graphs* and the *Janson dependency criterion* defined in Section 1.5.2. This method is used when we have a set of partially dependent random variables, for every value of n, and we want to prove that the sum of these variables has a certain asymptotic distribution. Assume that $\sigma := \underline{P}$ is a classical pattern and that $X_n := \mathrm{cnt}_\sigma(\pi)$ for $\pi \in S_n$ chosen uniformly at random. Then $X_n = \sum_{i=1}^{\binom{n}{k}} X_{n,i}$, where $X_{n,i}$ is an indicator, for every $i \in [\binom{n}{k}]$. In particular, if we fix some ordering of the $\binom{n}{k}$ size-k subsequences of π by the numbers in $[\binom{n}{k}]$, then for any $i \in [\binom{n}{k}]$, $X_{n,i} = 1$, if the subsequence of π with number i is an occurrence of the pattern σ. Otherwise, $X_{n,i} = 0$. Indeed, for any n, not all of the variables $X_{n,i}$ are independent.

Bóna applied the Janson criterion (Theorem 1.55) to the family $\left\{X_{n,i} \mid i = 1, 2, \ldots, \binom{n}{k}\right\}$. A main fact that he uses when checking the criterion is a lower bound on the variance of X_n. In this subsection, we reprove this lower bound by using Corollary 3.14 and Lemma 3.20 given below, which was established by Burstein and Hästö [33]. This gives a new proof that $\mathrm{cnt}_{\underline{P}}$ has asymptotically normal distribution, when $\underline{P}$ is a classical pattern. We also provide a new interpretation of Lemma 3.20.

Let $A_\sigma(r)$ denote the set of possible merges of two copies of the pattern σ, which is of size k, and where the resulting pattern is of size r. Formally, $A_\sigma(r)$ can be defined as the set of triples (π, m_1, m_2), such that $m_1, m_2 : \underline{\sigma}, \underline{\sigma} \to \underline{\pi}$ and $\pi \in S_r$. However, it will be more convenient for us to look at the subsequences of π formed by the images of the functions m_1 and m_2, i.e., we will use the following equivalent definition.

Definition 3.18. For $\sigma \in S_k$, let

$$A_\sigma(r) := \{(\underline{\pi}, x, y) \mid \pi \in S_r,\ x, y \in \mathrm{subs}(\pi),\ \mathrm{red}(x){=}\sigma,\ \mathrm{red}(y){=}\sigma,\ |x \cap y| = 2r - k\},$$

where $\mathrm{subs}(\pi)$ denotes the set of the subsequences of the permutation π.

Example 3.19. If $\underline{\sigma}$ is the classical pattern 312, then $A_{312}(5)$ is a set of triples containing $(54213, 523, 413)$, since $\mathrm{red}(523) = 312$, $\mathrm{red}(413) = 312$ and these two subsequences have exactly one common element (see Table III).

Let $a_\sigma(r) := |A_\sigma(r)|$.

Lemma 3.20 (Burstein and Hästö, [33, Lemma 4.3])**.** For any classical pattern $\sigma = \sigma_1 \cdots \sigma_k$,

$$a_\sigma(2k-1) > \binom{2k-1}{k}^2. \tag{3.11}$$

5	4	2	1	3
5		2		3
	4		1	3

TABLE III: Merge of two copies of the pattern 312.

Example 3.21. Let $k = 2$ and $\sigma = 21$. Then, $\binom{2k-1}{k}^2 = 9$ and $a_\sigma(2k-1) = a_{21}(3) = 10$, since $A_{21}(3)$ consists of the ten triples (π, x, y) given below:

$\pi = 321$: $(321, 32, 31)$, $(321, 31, 32)$, $(321, 32, 21)$, $(321, 21, 32)$, $(321, 31, 21)$, $(321, 21, 31)$.

$\pi = 312$: $(312, 31, 32)$, $(312, 32, 31)$.

$\pi = 231$: $(231, 21, 31)$, $(231, 31, 21)$.

Now, we are ready to prove the bound for the variance of cnt_σ used by Bóna.

Theorem 3.22. Let $X_n := \mathrm{cnt}_\sigma(\pi)$ be the number of occurrences of a classical pattern $\sigma \in S_k$ in a random permutation $\pi \in S_n$. Then, there exists $c > 0$, such that for all n,

$$\mathrm{Var}(X_n) \geq cn^{2k-1}.$$

Proof. Since σ is a classical pattern, Lemma 3.15 gives us that $\mathbb{E}(X_n) = \frac{\binom{n}{k}}{k!}$. Using this fact and Corollary 3.14, we obtain

$$\mathrm{Var}(X_n) = \mathbb{E}(X_n^2) - \mathbb{E}^2(X_n) = \left[a_\sigma(2k)\frac{\binom{n}{2k}}{(2k)!} + a_\sigma(2k-1)\frac{\binom{n}{2k-1}}{(2k-1)!} + O(n^{2k-2})\right] - \frac{\binom{n}{k}^2}{(k!)^2}.$$

We know that $\binom{n}{k} = \frac{(n)_k}{k!}$ and that $(n)_k = \sum_{i=0}^{k} s(k,i)n^i$, where $s(k,i)$ are the Stirling numbers of the first kind. We have $s(k,i) = (-1)^{k-i}\left[{k \atop i}\right]$, where $\left[{k \atop i}\right]$ is the number of permutations in S_k with i disjoint cycles. In particular, $\left[{k \atop k-1}\right] = \binom{k}{2}$. Therefore,

$$\mathrm{Var}(X_n) = [a_\sigma(2k)\tfrac{n^{2k}-\binom{2k}{2}n^{2k-1}}{((2k)!)^2} + a_\sigma(2k-1)\tfrac{n^{2k-1})}{((2k-1)!)^2}] - \tfrac{n^{2k}-2\binom{k}{2}n^{2k-1}}{(k!)^4} + O(n^{2k-2}).$$

It is easy to see that $a_\sigma(2k) = \binom{2k}{k}^2$ since a merge of size $2k$ of two copies of σ is uniquely determined by the set of k positions among $[2k]$, where the first copy will be placed, and the set of k values among $[2k]$ at these positions. The values and the positions for the letters of the second copy are those remaining. Then, one can see that the coefficient of $\mathrm{Var}(X_n)$ in front of n^{2k} is 0 and the coefficient in front of n^{2k-1} is

$$\frac{-\binom{2k}{k}^2\binom{2k}{2}}{((2k)!)^2} + \frac{a_\sigma(2k-1)}{((2k-1)!)^2} + \frac{2\binom{k}{2}}{(k!)^4}.$$

Simplify the last expression to get that this coefficient is positive, only if

$$a_\sigma(2k-1) > \binom{2k-1}{k}^2,$$

which follows from Lemma 3.20. □

It is interesting to note that Burstein and Hästö obtained the same bound for the variance of X_n in [33], but they did not state that it implies the central limit theorem for $\mathrm{cnt}_{\underline{P}}$, when $\underline{P}$ is an arbitrary classical pattern. At the same time, in [24], Bóna proved the bound independently and did not cite the work of Burstein and Hästö.

Next, we give an interpretation of Lemma 3.20, which may be useful to obtain a combinatorial proof for it. Let

$$A_{\sigma,\sigma'}(r) := \{(\pi, x, y) \mid \pi \in S_r,\ x, y \in \mathrm{subs}(\pi),\ \ \mathrm{red}(x){=}\sigma,\ \ \mathrm{red}(y){=}\sigma',\ |x \cap y| = 2r - k\},$$

be the set of merges of size r for the clssical patterns $\sigma \in S_k$ and $\sigma' \in S_k$. Let $a_{\sigma,\sigma'}(r) := |A_{\sigma,\sigma'}(r)|$.

Theorem 3.23. Lemma 3.20 is equivalent to

$$a_{\sigma}(2k-1) > \mathbb{E}(a_{\sigma,\sigma'}(2k-1)), \tag{3.12}$$

where $\sigma \in S_k$ is a fixed classical pattern and $\sigma' \in S_k$ is chosen uniformly at random.

Proof. First, note that $\binom{2k-1}{k}^2$, which is the right-hand side of Equation (3.11) in Lemma 3.20, can be written as $\frac{\binom{2k-1}{k}}{k}\binom{2k-1}{k}k$. Then, observe that $\binom{2k-1}{k}k$ is the number of ways to choose the k positions from $[2k-1]$ for the numbers of the subsequence x (that is order isomorphic to σ), as well as the position of the common element c for x and the subsequence y (that is order-isomorphic to σ'). For each of these choices, we can select the values of the numbers of x at the already selected positions in $\binom{2k-1}{k}$ ways. Once this choice is made, the values of x, y and c are uniquely determined. Suppose that c has to be at position p in y. Since σ' is chosen uniformly at random, we have probability $\frac{1}{k}$ for the element c to be at position p in y. This gives $\frac{\binom{2k-1}{k}}{k}$ for the expected number of merges when we know the positions of the elements of x and the position of c. Therefore,

$$\mathbb{E}(a_{\sigma,\sigma'}(2k-1)) = \frac{\binom{2k-1}{k}}{k}\binom{2k-1}{k}k = \binom{2k-1}{k}^2.$$

□

Interestingly, if σ is fixed, then $a_{\sigma,\sigma'}(k, 2k-1)$ does not necessarily reach its maximum when $\sigma' = \sigma$. For instance, $a_{1324,1234}(4,7) > a_{1324,1324}(4,7)$. However, since we know that Lemma 3.20 holds, Theorem 3.23 gives us that when $\sigma' = \sigma$, we always get a value greater than the expectation over σ'.

3.4.2 Vincular patterns

Recall that if $\boldsymbol{D}(\underline{P})$ is empty, then $\underline{P}$ is a vincular pattern and to denote it, we write P with the positions i and $i+1$ of P underlined, for every $i \in \boldsymbol{C}(\underline{P})$. The *blocks* of a vincular pattern are segments defined by $C(\underline{P})$. For example, if $\underline{P} = (135246, \{1,2,5\}, \emptyset) = \underline{135}2\underline{46}$, then $\boldsymbol{C}(\underline{P})$ has three blocks, namely 135, 2 and 46.

The limiting normality of $\mathrm{cnt}_{\underline{\sigma}}$, when $\underline{\sigma}$ is a vincular pattern was first established by Hofer [84]. She proposes two different approaches to bound the Kolmogorov distance between the distribution of $\mathrm{cnt}_{\underline{\sigma}}(\pi)$, for a randomly chosen $\pi \in S_n$, and a variable with a standard normal distribution. The Kolmogorov distance between two distributions was defined at the end of Section 1.5.2 togeter with a lemma that shows that bounding this distance is a sufficient condition for asymptotic normality. Both of these approaches are based on dependency graphs. To apply them, Hofer needs a lower bound for the variance of $\mathrm{cnt}_{\underline{\sigma}}$, i.e., to prove a more general version of Theorem 3.22, which holds for any vincular pattern. Hofer obtained such a generalization by a rather complicated recurrence based on the law of total variance.

Theorem 3.24 (Hofer [84])**.** Let $X_n = \mathrm{cnt}_{\underline{\sigma}}$ be the number of occurrences of a vincular pattern $\underline{\sigma}$ with j blocks, in a random permutation of size n. Then, there exists $c > 0$, such that for all n,

$$\mathrm{Var}(X_n) \geq cn^{2j-1}.$$

Below, we show that this more general bound is equivalent to a lemma generalizing Lemma 3.20 that has an analogous interpretation to the one given by Theorem 3.23.

If $\underline{\sigma}$ is a vincular pattern of size k with j blocks, then we denote by $b_{\underline{\sigma}}(m, j')$ the number of merges of two copies of $\underline{\sigma}$, where the resulting pattern is of size m and has j' blocks.

Example 3.25 (merge of two copies of a vincular pattern)**.** Let $\underline{\sigma} = \underline{431}\,\underline{52}$. This pattern has size $k = 5$ and $j = 2$ blocks. Table Table IV gives an example of a merge of two copies of $\underline{\sigma}$. The resulting pattern $\underline{6531}\,\underline{84}\,\underline{72}$ is of size $m = 8$ and has $j' = 3$ blocks.

6	5	3	1	8	4	7	2
6	5	3		8	4		
	5	3	1			7	2

TABLE IV: Merge of two copies of the pattern $\underline{431}\,\underline{52}$.

If $\underline{\sigma}$ has blocks of sizes $\alpha_1, \ldots, \alpha_j$, then let $M_{\underline{\sigma}} = \max_{1 \le i \le j} \{\alpha_i\}$ and let $[x^k]P$ denote the coefficient of the polynomial P in front of x^k.

Theorem 3.26. Theorem 3.24 is equivalent to

$$\sum_{l=1}^{M_{\underline{\sigma}}} (2k)_l b_{\sigma}(2k-l, 2j-1) > \binom{2k}{k}\binom{2j-1}{j} j. \tag{3.13}$$

Proof. We will use that the expected number of occurrences of a vincular pattern $\underline{\sigma}$ of size k, with j blocks, in a random permutation of size n is $\frac{\binom{n-(k-j)}{j}}{k!}$. This follows from Lemma 3.15 and the fact that $|\boldsymbol{C}(\underline{\sigma})| = k - j$. Apply Corollary 3.14 and note that if $m_1, m_2 : \underline{\sigma}, \underline{\sigma} \to \underline{P}$ and $\underline{P}$ has $2j-1$ blocks, then exactly one block of the first copy of $\underline{\sigma}$ was merged with one block of the second copy of $\underline{\sigma}$. Therefore, $|P| = \tilde{k} \in [2k - M_{\underline{\sigma}}, 2k]$ and $|\boldsymbol{C}(\underline{P})| = \tilde{k} - (2j-1)$. We have

$$\begin{aligned} \mathrm{Var}(X_n) = \mathbb{E}(X_n^2) - \mathbb{E}^2(X_n) = &\left[b(2k, 2j)\frac{\binom{n-(2k-2j)}{2j}}{(2k)!} + \sum_{l=1}^{M_{\underline{\sigma}}} b_{\sigma}(2k-l, 2j-1)\frac{\binom{n-(2k-l-2j+1)}{2j-1}}{(2k-l)!} \right] \\ &- \frac{\binom{n-(k-j)}{j}^2}{(k!)^2} + O(n^{2j-2}). \end{aligned}$$

We will again use that $\binom{n}{k} = \frac{(n)_k}{k!}$ and that $(n)_k = \sum_{i=0}^{k} s(k,i) n^i$, where $s(k,i) = (-1)^{k-i} \left[{k \atop i} \right]$ are the Stirling numbers of the first kind and $\left[{k \atop i} \right]$ is the number of permutations in S_k with i disjoint cycles. Since $\left[{k \atop k-1} \right] = \binom{k}{2}$ and $b_{\sigma}(2k, 2j) = \binom{2k}{k}\binom{2j}{j} = \frac{(2j)!(2k)!}{(k!j!)^2}$, we get the following.

$$\begin{aligned}
\mathrm{Var}(X_n) &= \left[\frac{(2j)!(2k)!}{(k!j!)^2}\frac{(n-2k+2j)_{2j}}{(2j)!(2k)!} + \sum_{l=1}^{M_{\underline{\sigma}}} b_\sigma(2k-l,2j-1)\frac{(n-(2k-l-2j+1))_{2j-1}}{(2k-l)!(2j-1)!}\right] - \frac{(n-k+j)_j^2}{(k!)^2(j!)^2} + O(n^{2j-2})\\
&= \frac{1}{(k!j!)^2}[(n-2k+2j)^{2j} - \binom{2j}{2}(n-2k+2j)^{2j-1} - ((n-k+j)^j - \binom{j}{2}(n-k+j)^{j-1} + O(n^{j-2}))^2]\\
&+ \sum_{l=1}^{M_{\underline{\sigma}}} b_\sigma(2k-l,2j-1)\frac{(n-2k+l+2j-1)^{2j-1}+O(n^{2j-2})}{(2k-l)!(2j-1)!} + O(n^{2j-2})\\
&= \frac{1}{(k!j!)^2}[n^{2j} - (2k-2j)n^{2j-1} - j(2j-1)n^{2j-1} - (n^j - (k-j)n^{j-1} - \frac{j(j-1)}{2}n^{j-1} + O(n^{j-2}))^2]\\
&+ \sum_{l=1}^{M_{\underline{\sigma}}} b_\sigma(2k-l,2j-1)\frac{n^{2j-1}}{(2j-1)!(2k-l)!} + O(n^{2j-2}).
\end{aligned}$$

After simplifying, we get that $[n^{2j-1}]\,\mathrm{Var}(X_n) > 0$ if and only if

$$\begin{aligned}
&\frac{-j^2}{(k!j!)^2} + \sum_{l=1}^{M_{\underline{\sigma}}} \frac{b_\sigma(2k-l,2j-1)}{(2k-l)!(2j-1)!} > 0 \iff\\
&\sum_{l=1}^{M_{\underline{\sigma}}}(2k)_l b_\sigma(2k-l,2j-1) > \binom{2k}{k}\binom{2j-1}{j}j.
\end{aligned}$$

□

Note that when $j = k$, we have $M_{\underline{\sigma}} = 1$ and $b_\sigma(2k-1,2j-1) = a_\sigma(2k-1)$, so we get Lemma 3.20. When $j = 1$, Equation (3.13) is trivial, since $M_{\underline{\sigma}} = k$ and on the left, just one of the summands (when $l = k$) is $(2k)_k$, while on the right we have $\binom{2k}{k} < (2k)_k$.

We were not able to prove Equation (3.13) for vincular patterns with arbitrary number of blocks. However, we can give an interpretation of this inequality. Note that when one merges two copies of a pattern with j blocks and the obtained pattern has $2j-1$ blocks, then the blocks of the two copies can be aligned in exactly $\binom{2j-1}{j}j$ ways. These alignments will be called *configurations*. For example, when $j = 2$, there are $\binom{3}{2}2 = 6$ configurations shown below (the ■ symbol represents a block):

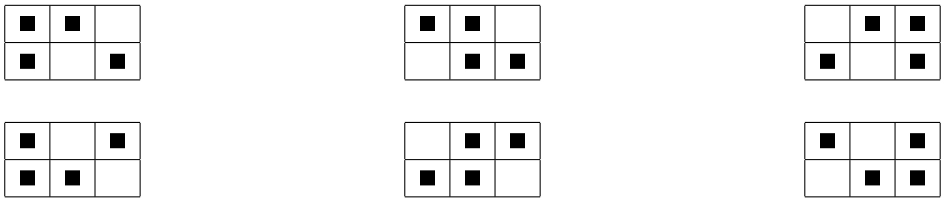


Figure 15: The 6 possible configurations, when merging two copies of a pattern with two blocks.

For instance, the configuration corresponding to the merge shown in Table IV is the top-left configuration shown on Figure 15. It is not difficult to see that the conjecture we give next would imply Equation (3.13) and respectively Theorem 3.24 and the CLT for vincular patterns.

Conjecture 3.27. For every vincular pattern σ with j blocks and every $1 \leq l \leq M_{\underline{\sigma}}$,

$$b'_{\sigma}(2k-l, 2j-1) > \frac{\binom{2k-l}{k}}{k_{(l)}} c_{\sigma,l}, \tag{3.14}$$

where $c_{\sigma,l} :=$ is the number of possible configurations for a merge of two copies of $\underline{\sigma}$, such that the minimum of the sizes of the two merged blocks is l, and $b'_{\sigma}(2k-l, 2j-1)$ is the number of merges of two copies of $\underline{\sigma}$ with l common elements and $2j-1$ blocks, such that they correspond to one of the same $c_{\sigma,l}$ configurations.

Indeed, it suffices to note that $\sum_{l=1}^{M_{\sigma}} c_{\sigma,l} = \binom{2j-1}{j} j$ and that $\frac{\binom{2k-l}{k}}{k_{(l)}} = \frac{\binom{2k}{k}}{(2k)_l}$. Thus, if we sum Equation (3.14) over l, we get Equation (3.13) with b_{σ} replaced with b'_{σ}. Since $b_{\sigma}(2k-l, 2j-1) \geq b'_{\sigma}(2k-l, 2j-1)$, for all l, j and k, Conjecture 3.27 would indeed imply Equation (3.13). The ratio $\frac{\binom{2k-l}{k}}{k_{(l)}}$ is the expected number of merges when we fix one of the $c_{\sigma,l}$ configurations and when we merge $\underline{\sigma}$ and $\underline{\sigma'}$, where $\sigma' \in S_k$ is a permutation selected uniformly at random and $\underline{\sigma'}$ has the same block structure as $\underline{\sigma}$. Therefore, Equation (3.14) can be written as

$$b'_{\sigma}(2k-s,2j-1) > \mathbb{E}(b'_{\sigma,\sigma'}(2k-s,2j-1)), \tag{3.15}$$

where $b'_{\sigma,\sigma'}(2k-s,2j-1)$ is defined analogously to $a_{\sigma,\sigma'}(2k-1)$.

3.4.3 Bivincular patterns

In the general case when $\underline{P}$ is a pattern for which $\boldsymbol{D}(\underline{P})$ might be non-empty, we do not necessarily have asymptotic normality of the distribution of $\mathrm{cnt}_{\underline{P}}$. For example, the adjacency statistic adj introduced in Section 3.3 and corresponding to the bivincular pattern $(21,\{1\},\{1\})$, has Poisson distribution with mean 1. This follows from a result proved, independently by Wolfowitz [143] and Kaplansky [94] in the 1940s. They showed that if X denotes the pairs of numbers a, $a+1$ that have consecutive positions in a permutation in S_n that is chosen uniformly at random, then X is asymptotically Poisson distributed with mean 2. In 2014, Corteel et al. [40] give another proof of this result that uses the method of Chen, which is used to prove convergence to Poisson distribution and which is an adaptation of the method of Stein for convergence to normal distribution [136]. The article [7] contains an accessible introduction and good examples.

Here, we reprove the fact that the asymptotic distribution of adj is Poisson with mean 1 by using Theorem 3.17 and the Fréchet-Shohat Theorem given below.

Theorem 3.28 ([21, Theorem 30.2]). Suppose that the distribution of X is determined by its moments and that X_n have moments of all orders. Suppose also that $\lim_{n\to\infty}\mathbb{E}(X_n^r) = \mathbb{E}(X^r)$, for $r = 1, 2, \ldots$. Then, X_n converges in distribution to X.

Definition 3.29. The discrete random variable $X := \mathrm{Po}(\lambda)$ is said to have a Poisson distribution, with parameter $\lambda > 0$, if

$$\mathbb{P}(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \qquad \text{for } k = 0, 1, \ldots \cdot$$

Theorem 3.30. As $n \to \infty$, adj converges in distribution to $\mathrm{Po}(1)$.

Proof. The Poisson measure is determined by its moments [21, Theorem 30.1]. Because of Theorem 3.28, it suffices to show that $\mathbb{E}(\mathrm{adj}^r)$ converges to the r-th moment of $Po(1)$, when $n \to \infty$. A well-known fact is that the r-th moment of the Poisson distribution with mean 1 is the r-th Bell number $B_r = \sum_{k=1}^{r} S(r,k)$, where $S(r,k)$ is the Stirling number of the second kind (for more details, see [122]). Looking at the double sum expression for $\mathbb{E}(\mathrm{adj}^r)$ obtained in Theorem 3.17, we see that every summand is a product of terms not including n and the term $\frac{(\binom{n-(m-u)}{u})^2}{n_{(m)}}$ is $O(1)$, unless $u = \frac{m}{2}$. Thus, when $n \to \infty$, we can look only at the terms corresponding to even values of m, i.e., $m = 2m_1$ for some $m_1 = 1, \ldots, r$ and $u = \frac{m}{2} = m_1$. Since $\lim_{n\to\infty} \frac{\binom{n-m_1}{m_1}^2}{n_{(2m_1)}} = \frac{1}{(m_1!)^2}$ and $\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^r = k!S(r,k)$, we obtain the following.

$$\lim_{n\to\infty}\mathbb{E}(\mathrm{adj}^r) = \sum_{m_1=1}^{r}\left(\sum_{w=0}^{m_1}(-1)^w\binom{m_1}{w}(m_1-w)^r\right)m_1!\frac{1}{(m_1!)^2} =$$
$$= \sum_{m_1=1}^{r}\frac{\sum_{w=0}^{m_1}(-1)^w\binom{m_1}{w}(m_1-w)^r}{m_1!} = \sum_{m_1=1}^{r} S(r,m_1) = B_r. \qquad \square$$

CHAPTER 4

PERMUTATION PATTERNS WITH CONSTRAINED GAP SIZES

In this chapter, we investigate some facts related to the distant patterns(DPs) introduced in Section 1.6. Recall that DPs allow arbitrary minimum requirements for the size of the gap between the numbers in the permutation, corresponding to two consecutive letters of the pattern.

4.1 Two basic facts about distant patterns

Avoidance of classical distant patterns can be formulated as a statement about simultaneous avoidance of classical patterns. For example, avoiding $1\square 2$ is equivalent to the simultaneous avoidance of the 3-letter classical patterns $\{123, 132, 213\}$. In the general case, we have the proposition below, where

$$x^{(y)} := x(x-1)\cdots(x-y+1) = \tfrac{x!}{(x-y)!}$$

denotes the falling factorial. We leave the formal proof of this proposition to the reader.

Proposition 4.1. The avoidance of $q = \square^{r_0} q_1 \square^{r_1} q_2 \square^{r_2} \cdots \square^{r_{k-1}} q_k \square^{r_k}$, where $\sum_{j=0}^{k} r_j = S$, is equivalent to the simultaneous avoidance of $(S+k)^{(S)}$ classical patterns of size $S+k$.

In fact, we can restrict our attention to DPs without leading or trailing squares.

Theorem 4.2. For any $r_1, r_2 > 0$ and a distant pattern q, we have

$$av_n(\square^{r_1} q \square^{r_2}) = n^{(r)} av_{n-r}(q), \tag{4.1}$$

where $r := r_1 + r_2$.

Proof. If $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathrm{Av}_n(\square^{r_1}q\square^{r_2})$, then

$$\mathrm{red}(\sigma_{r_1+1}\cdots\sigma_{n-r_2}) \in \mathrm{Av}_{n-r}(q).$$

Conversely, any $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ for which $\sigma_1,\ldots,\sigma_{r_1},\sigma_{n-r_2},\ldots,\sigma_n$ are any r numbers in $[n]$ and for which $\mathrm{red}(\sigma_{r_1+1}\cdots\sigma_{n-r_2}) \in \mathrm{Av}_{n-r}(q)$, would be such that $\sigma \in \mathrm{Av}_n(\square^{r_1}q\square^{r_2})$, since any possible occurrence of q in σ would have either fewer than r_1 other elements in front of it or fewer than r_2 elements after it. $\square$

4.2 Classical DPs of size 2

By Theorem 4.2, it suffices to consider $1\square2$ and $2\square1$ as the only DPs of size 2. One can obviously see that $av_n(2\square1) = av_n(1\square2)$, by applying the reverse map. Therefore we have only one Wilf-equivalence class here. The enumeration of this class appears in the seminal paper of Simion and Schmidt [129], where they showed that $av_n(123, 132, 213) = F_{n+1}$ (see Theorem 4.1). Below, we present a proof in the language of DPs.

Theorem 4.3 (Simion and Schmidt [129])**.** For $n \geq 3$,

$$av_n(2\square1) = F_{n+1}, \tag{4.2}$$

i.e., the $(n+1)$st Fibonacci number.

Proof. If $p_1p_2\cdots p_n \in \mathrm{Av}_n(2\square1)$, then either $p_n = n$ or $p_{n-1} = n$ since otherwise n will participate in an inversion that is not of consecutive letters. If $p_n = n$ then $p_1p_2\cdots p_{n-1}$ must be in $\mathrm{Av}_{n-1}(2\square1)$, and for each permutation in $\mathrm{Av}_{n-1}(2\square 1)$, we obtain a permutation in $\mathrm{Av}_n(2\square 1)$ after appending the letter n at the end. Thus, we have $av_{n-1}(2\square 1)$ permutations in $\mathrm{Av}_n(2\square 1)$ ending with n. If $p_{n-1} = n$, then we must have $p_n = n-1$ to prevent $n-1$ from forming a prohibited inversion with p_n. Thus, in this second case we must have $p_{n-1} = n$ and $p_n = n-1$ and the prefix $p_1 p_2 \cdots p_{n-2}$ must be a permutation in $\mathrm{Av}_{n-2}(2\square 1)$. For each such permutation in $A_{n-2}(2\square 1)$, we obtain a new one in $\mathrm{Av}_n(2\square 1)$ by appending n and then $n-1$ to it. Therefore $av_n(2\square 1) = av_{n-1}(2\square 1) + av_{n-2}(2\square 1)$. It remains to note that $av_3(2\square 1) = 3$ and $av_4(2\square 1) = 5$. $\square$

One can consider more general settings for DPs and look at bigger values of the maximal distance between two consecutive letters of a pattern. Recall that $\mathrm{Av}_n(2\square^r 1)$ is the set of all $p_1 p_2 \cdots p_n \in S_n$ with no inversion (p_i, p_j), such that $|i-j| > r$. Obviously, $av_n(2\square^m 1) = n!$ for $n \leq m+1$ and $av_n(2\square^0 1) = av_n(21) = 1$. The theorem below addresses the general case.

Theorem 4.4. The permutations in $\mathrm{Av}_n(2\square^r 1)$ are in one-to-one correspondence with the permutations in S_n for which, when written in a cycle notation, any two elements in a cycle differ by at most r.

Proof. Let $X = \mathrm{Av}_n(2\square^r 1)$ be the set of all permutations in S_n that do not have inversions at distance greater than r in their one-line notation. Let Y be the set of those permutations in S_n for which any two elements in the same cycle differ by at most r. We will describe a bijective map $f : X \to Y$. Consider $p = p_1 p_2 \cdots p_n \in X$. We will show how to obtain the standard form of $f(p)$ written in cycle notation, i.e., the minimal element of every cycle is at its first position and the cycles are ordered in increasing order of their minimal elements. Below is the description of f.

The number p_1 is at position 1, so let us look at the set of positions of all numbers with which p_1 is in inversion: $1, 2, \cdots, p_1 - 1$. Denote their positions with $i_1, i_2, \cdots, i_{p_1-1}$, respectively. These positions are not bigger than $r + 1$, since $p \in X$. Then take $(1, i_{p_1-1}, i_{p_1-2}, \cdots, i_1)$ to be the first cycle in the standard form of the cycle decomposition for $f(p)$. Then, let j be the minimal number that is not already used in this cycle decomposition, and let the numbers $p_j - 1, \cdots, p_1 + 1$ be at positions $j_{p_j-1}, j_{p_j-2}, \cdots, j_{p_1+1}$. Take $(j, j_{p_j-1}, j_{p_j-2}, \cdots, j_{p_1+1})$ as the next cycle in the standard form of the cycle decomposition for $f(p)$ and continue in the same way afterwards. Note that the length of some of those cycles might be 1.

Here are two examples:

- If $n = 9, r = 3$ and $p = 352149867$, then $f(352149867) = (134)(25)(6798)$.
- If $n = 8, r = 4$ and $p = 41352867$, then $f(41352867) = (1352)(4)(687)$.

Obviously, f maps each $\sigma \in X$ to a permutation $f(\sigma)$ such that any two numbers in the same cycle of $f(\sigma)$ differ by at most r, since these two numbers correspond to indices of two numbers, in the one-line notation of σ, which are in inversion in $\sigma \in X$. To prove that f is indeed a bijection, we will describe its inverse. Consider $\pi \in Y$ in its standard cycle decomposition form. If the first cycle of π is $(\pi_1\pi_2 \cdots \pi_{i_1})$, then put the number i_1 in the first place, then $i_1 - 1$ at position π_2, $i_1 - 2$ at position π_3 and so on. The number 1 will be placed at position π_{i_1}. Note also that π_1 is always 1. Next, go to the next cycle $(\pi_{j_1}\pi_{j_2} \cdots \pi_{j_{i_2}})$. We will determine the positions of the next i_2 numbers: $i_1 + 1, i_1 + 2, \cdots, i_1 + i_2$. We can see that π_{j_1} must be the least integer not occurring in the first given cycle. We will place at this position, the number $i_1 + i_2$. Then, $i_1 + i_2 - 1$ should be placed at position π_{j_2}, $i_1 + i_2 - 2$ at position π_{j_3} and so on. One can use the two given examples above for verification. $\square$

The sequences of the numbers $av_n(2\square^r 1)$, for different fixed values of n and r are respectively rows and columns of the table described in [118, A276837].

4.2.1 Recurrence formula for $F_{n+1} = av_n(1\square 2)$

While trying to obtain a general enumeration approach when dealing with DPs, we used a technique which helped us to obtain a recurrence formula for the number of permutations avoiding the distant pattern $1\square 2$, i.e., a new recurrence formula for the Fibonacci numbers (see Theorem 4.3). We describe this result in the current section. The same technique can be used when trying to enumerate the set of avoiders for other DPs.

The idea is that almost all permutations containing a given distant pattern can be obtained by first taking a permutation containing the corresponding classical pattern and then inserting additional numbers between some of the letters (where we have the $\square$ symbol) for a certain occurrence of this classical pattern. Let us describe this more concretely with the following algorithm that we will use for the pattern $1\square 2$. Recall that $C_m(p) := S_m \setminus \mathrm{Av}_m(p)$, for any pattern p.

Algorithm 1

1. For a given $n \geq 3$ and $j \in [n]$, take any $\pi \in C_{n-1}(12)$.
2. Find the leftmost 1 that is part of a classical 12-pattern and insert the number j immediately after it.
3. Increase by 1 the numbers $j, j+1, \ldots, n-1$, except the j that we just inserted (unless $j = n$, π contains another j).

This algorithm defines a map $g : A_{n-1} \to B_n$, where $A_{n-1} := C_{n-1}(12) \times [n]$ and $B_n := C_n(1\square 2)$.

Example 4.5. $g(3412, 2) = 42513$. The leftmost occurrence of the pattern 12 in 3412 is by the first two letters, 3 and 4. Therefore we insert $j = 2$ immediately after the letter 3 and then increase the $2, 3$ and 4 in the original permutation. Note that the added number j always keep its value in the final image.

We will first need to show that this map is not far from being injective.

Theorem 4.6. No permutation in $B_n = C_n(1\square 2)$, the range of the map g, is the image of more than two different elements of $A_{n-1} = C_{n-1}(12) \times [n]$.

Proof. Assume the opposite. Let $\pi = g(\pi_1, j_1) = g(\pi_2, j_2) = g(\pi_3, j_3)$ for three different tuples $(\pi_1, j_1), (\pi_2, j_2), (\pi_3, j_3) \in A_n = C_{n-1}(12) \times [n]$. We can see that j_1, j_2 and j_3 must be different since if two of them, say j_1 and j_2, are equal then obviously $\pi_1 = \pi_2$ and we will not have different tuples. Now, we know that without loss of generality $1 < c_{j_1} < c_{j_2} < c_{j_3}$ are three different positions for the three different numbers j_1, j_2, j_3 in the final permutation π. By step 2 of Algorithm 1, after removing j_1 from π, the first occurrence of the classical pattern 12, should be some $\pi_x\pi_y$, where $x = c_{j_1} - 1$. Similarly, after removing j_2, the first such occurrence should begin at position $c_{j_2} - 1 > c_{j_1} - 1$, but this is only possible if the position $y = c_{j_2}$ since if this is not the case then $\pi_x\pi_y$ would be an occurrence of 12 that begins before position $c_{j_2} - 1$. However, after removing j_3 from π (note that $c_{j_3} > c_{j_2}$), the first occurrence of 12 should begin at position $c_{j_3} - 1 > c_{j_1} - 1$. This is a contradiction. □

There are many permutations in B_n which are the image of g for two different elements of A_{n-1}. For example, $g(312, 4) = g(231, 1) = 3142$. The next fact that we will need gives the number of these permutations.

Theorem 4.7. The number of permutations in B_n which are an image of exactly two different elements of A_{n-1}, after applying the map g, is given by the sum

$$\sum_{j=3}^{n-1}(j-2)(n-j)(n-j)!. \tag{4.3}$$

Proof. Let $\pi = g(\pi_1, x) = g(\pi_2, y)$ for $\pi_1, \pi_2 \in C_{n-1}(12)$ and $x, y \in [n]$, where $(\pi_1, x) \neq (\pi_2, y)$ are different. We saw in the proof of Theorem 4.6 that x and y must be different. Let us denote the positions of x and y in π by i and j, respectively. Without loss of generality, let $i < j$. We know that after removing $y = \pi_j$ from π, then $\pi_{j-1}\pi_{j+k}$, for some $k \geq 1$, is the first occurrence of the classical pattern 12. Note that some of the two letters, π_{j-1} and π_{j+k}, may have decreased in value by one. Therefore, we should have $\pi_1 > \pi_2 > \cdots > \pi_{j-1}$. Since, if we remove $x = \pi_i$ from π, then $\pi_{i-1}\pi_j$ must be the first occurrence of 12, it follows that we must have $\pi_1 > \pi_2 > \cdots > \pi_{i-2} > \pi_j > \pi_{i-1} > \cdots > \pi_{j-1}$. In other words, the number π_j is between π_{i-2} and π_{i-1}. Otherwise, we would have a $12-$occurrence ending at π_j that starts before position $i-1$. We also have that $\pi_{j-t} > \pi_{j+l}$, for any $t = 2, \cdots, j-1$ and $l = 1, \cdots, n-j$, because otherwise when removing π_j from π, a 12-occurrence starting before π_{j-1} will be present.

In order to determine π completely, we must known the relations between the $n-j+1$ numbers $\pi_{j-1}, \pi_{j+1}, \pi_{j+2}, \cdots, \pi_n$. The only constraint that we have is that π_{j-1} is not the largest among them. Thus, when i and j are fixed, we always have $(n-j+1)! - (n-j)!$ possible ways to write π. Therefore, the number of different permutations $\pi \in S_n$ that are an image for two different tuples is

$$\sum_{j=3}^{n-1}\sum_{i=2}^{j-1}[(n-j+1)! - (n-j)!] = \sum_{j=3}^{n-1}(j-2)(n-j)(n-j)!.$$

Each term in the latter sum gives us the number of permutations in $\pi \in B_n$, where $\pi = g(\pi_1, x) = g(\pi_2, y)$ for some $\pi_1, \pi_2 \in C_{n-1}(12)$ and $x, y \in [n]$, where $x < y$ and y is at position j in π. $\square$

As we have seen that no permutation in B_n is counted more than two times, it remains to obtain the number of permutations in B_n that are not an image of g for any permutation in the set of tuples A_{n-1}.

Theorem 4.8. The number of permutations in the set $B_n \setminus g(A_{n-1})$ is:

$$\sum_{k=3}^{n-2} (F_{n-k+1} - 1)k(k-2)(k-2)!, \tag{4.4}$$

where F_i denotes the i-th Fibonacci number.

Proof. An example of a permutation in B_n that cannot be obtained as an output of the function g (i.e., with Algorithm 1) for any input in A_{n-1} is the permutation 45132. The reason is that before the first occurrence of the distant pattern $1\square 2$, there exist an occurrence of the classical pattern 12. We want to obtain a formula for all permutations in B_n having this property. Consider one such permutation $\pi = \pi_1\pi_2 \cdots \pi_n$ and let the first occurrence of $1\square 2$ be $\pi_j\pi_{j+v}$ for some $j \geq 1$, $v \geq 2$, and $j + v \leq n$. Since, this is the first such occurrence, observe that $\pi_i > \pi_{j+d}$, for any $i < j$ and $d \geq 0$. Otherwise we would have another occurrence of $1\square 2$, preceding $\pi_j\pi_{j+v}$. Thus $\pi' = \pi_1 \cdots \pi_{j-1}$ must be a permutation of the $j-1$ numbers in $[n-j+2, n]$ and $\pi'' = \pi_j \cdots \pi_n$ is simply a permutation of $[1, n-j+1]$ for which $\pi_j < \pi_{j+v}$ for some $v = 1 \cdots n-j$. This means that π' avoids $1\square 2$, but contains the classical pattern 12. Therefore the number of possibilities for π' is $av_{j-1}(1\square 2) - av_{j-1}(12) = F_j - 1$ since $av_{j-1}(1\square 2) = F_j$ (Theorem 4.3), $av_{j-1}(12) = 1$ and $A_{j-1}(12) \subset A_{j-1}(1\square 2)$. Now, let us denote $k = n - j + 1$, for clarity. For π'', we can see that it could be any k-permutation except that it could not start with k or with $(k-1)k$

since if this is the case, then π'' will not start with an occurrence of the $1\square 2$ pattern. The latter means that the possible values for π'' are exactly $k! - (k-1)! - (k-2)! = k(k-2)(k-2)!$. Summing over k, we obtain the given formula. $\square$

Now, we are ready to derive the recurrence formula that we want. Note that $|A_{n-1}| = |C_{n-1}(12) \times [n]| = ((n-1)! - 1) \cdot n = n! - n$ gives the number of permutations in B_n that are the image of the map g for exactly one tuple in A_{n-1}. Theorems 4.7 and 4.8 give the number of permutations being the image of g for 2 and 0 tuples in A_{n-1}, respectively. We also know that $|B_n| = |C_n(1\square 2)| = n! - F_{n+1}$. Thus using inclusion-exclusion we have:

$$n! - F_{n+1} = (n! - n) - \sum_{j=3}^{n-1} (j-2)(n-j)(n-j)! + \sum_{k=3}^{n-2} (F_{n-k+1} - 1)k(k-2)((k-2)!).$$

After simplifying, we obtain the following recurrence formula for the Fibonacci numbers and respectively for the number of permutations avoiding the distant pattern $1\square 2$ (or $2\square 1$):

$$av_n(1\square 2) = F_{n+1} = n + \sum_{k=1}^{n-3} (n - (k+2)F_{n-(k+1)}) \cdot k \cdot k!. \tag{4.5}$$

4.3 Classical DPs of size 3

As we can infer from Theorem 4.1, finding a closed formula for the avoidance set of a distant pattern becomes more complicated as its size increases, because the number of classical patterns that must be simultaneously avoided increases, as well. In this section, we describe some already established results on the DPs of size 3 with one square ($xy\Box z$) and two squares ($x\Box y\Box z$). Then, we discuss an approach that can be used to obtain the generating function for $av_n(1\Box 3\Box 2)$.

4.3.1 Patterns of the kind $xy\Box z$

Consider the patterns $xy\Box z$ and $x\Box yz$, for some permutation $xyz \in S_3$. The thesis of Firro [70] and two related works [71, 72] give the formula

$$av_n(12\Box 3) = \sum_{k\geq 0} \frac{1}{n-k}\binom{2n-2k}{n-1-2k}\binom{n-k}{k}. \tag{4.6}$$

The same thesis gives two bijections between $12\Box 3$-avoiding permutations and odd-dissections of a given $(n+2)$-gon, which are dissections with non-crossing diagonals so that no $2m$-gons ($m > 1$) appear [118, A049124]. In fact, it turns out that this is the cardinality of the avoidance set for any pattern of the kind $xy\Box z$ or $x\Box yz$ [70]. We know that all classical patterns in S_3 are avoided by the same number of permutations, namely the Catalan numbers. One might suspect that whenever two classical patterns $p, q \in S_k$ are Wilf-equivalent, then inserting a square at the same place in p and q will produce two Wilf-equivalent distant patterns. The computer simulations suggest that this is true for the Wilf-equivalent patterns $\{1234, 1243, 2143\}$. We have formulated this conjecture in Section 5.3.

It was shown in [70] that if $xyz \in S_3$, then inserting a square between x and y or between y and z always gives us two Wilf-equivalent patterns. It is worth noting that we do not have a similar fact when considering patterns of size four. For example, $av_7(1\Box 234) = 3612 \neq 3614 = av_7(12\Box 34)$.

4.3.2 Patterns of the kind $x\square y\square z$

The inverse and the complement map give us at most two Wilf-equivalent permutation classes for these patterns: $\{\text{dist}_1(p) \mid p = 132, 231, 213, 312\}$ and $\{\text{dist}_1(p) \mid p = 123, 312\}$. Unlike the case of classical patterns in which these are, in fact, one class [129], here, these classes are different.

Theorem 4.9 (Hopkins and Weiler [85])**.** For $n > 5$,

$$av_n(\text{dist}_1(123)) > av_n(\text{dist}_1(132)). \tag{4.7}$$

The theorem above is a special case of a result of Hopkins and Weiler [85, Theorem 3]. In that work they extend the result of Simion and Schmidt that $av_n(123) = av_n(132)$ from permutations on a totally ordered set to a similar result for pattern avoidance in permutations on partially ordered sets. In particular, they show that $av_{P,n}(132) \leq av_{P,n}(123)$ for any poset P, where $\text{Av}_{P,n}(q)$ is the number of n-permutations on the poset P avoiding the pattern q. Furthermore, they classify the posets for which equality holds. Here, we state the corollary of their result generalizing Theorem 4.9, as formulated by the authors.

Theorem 4.10 (Hopkins and Weiler [85])**.** For $r \geq 0$ and $n \geq 1$, we have

$$av_n(\text{dist}_r(123)) \geq av_n(\text{dist}_r(132)), \tag{4.8}$$

with strict inequality if and only if $r \geq 1$ and $n \geq 2r + 4$.

Note that in the case $n = 2r + 3$, $av_{2r+3}(\mathrm{dist}_r(123)) = av_{2r+3}(\mathrm{dist}_r(132))$ since there is only one triple of positions where each of these two patterns can occur in a $(2r + 3)$-permutation, namely the positions $1, r + 2$ and $2r + 3$. So for each such occurrence, we can exchange the elements at positions $r+2$ and $2r+3$ to get an occurrence of the other pattern. A similar statement about consecutive patterns was first proved in [57] with a simple injection. It states that $av_n(\underline{123}) > av_n(\underline{132})$ for every $n \geq 4$. The listed facts imply that the monotonic pattern 123 is avoided more frequently than 132 when we have two gaps of size exactly 0 between the letters in each occurrence of the two patterns, or when the minimal constraint for each gap is some fixed positive number. However, when patterns with all possible gap sizes must be avoided, we have an equality since $av_n(123) = av_n(132)$. We address this surprising fact in the next section.

Along those lines is another work of Elizalde [56] on consecutive patterns, where he generalizes [57] by proving that the number of permutations avoiding the monotone consecutive pattern $\underline{12\cdots m}$ is asymptotically larger than the number of permutations avoiding any other consecutive pattern of size m. He also proved there that $av_n(\underline{12\ldots(m-2)m(m-1)})$ is asymptotically smaller than the number of permutations avoiding any other consecutive pattern of the same size. Similar conjectures can be formulated for DPs (see Section 5.3).

4.3.2.1 The pattern $1\square 3\square 2$

In this subsection, we will describe an approach that one can use to find the generating function $G(x) = \sum_{n\geq 0} av_n(q)x^n$, where $q = 1\square 3\square 2 = \mathrm{dist}_1(132)$. The complete proof is rather technical, so we omit many details. We will need to define the following sets of permutations:

$$\mathbb{H}_1 := \{\pi \mid \pi \in \mathrm{Av}(q),\ |\pi| \geq 1 \text{ and } \pi \text{ has no occurrence of } 1\square\underline{32} \text{ ending at the last position of } \pi\},$$

$$\mathbb{H}_2 := \{\pi \mid \pi \in \mathrm{Av}(q),\ |\pi| \geq 1 \text{ and } \pi \text{ has no occurrence of } \underline{13}\square 2 \text{ beginning at the first position of } \pi\}.$$

Let us also denote the corresponding generating functions with

$$H_i(x) = \sum_{k=1}^{\infty} h_i(k)x^k,$$

where $h_i(n)$ is the number of permutations of size n in $\mathbb{H}_i$. Now, we can describe a useful decomposition for the permutations in $\mathrm{Av}_n(q)$ which is similar, but more complicated, to the one given in [70] for the permutations in $\mathrm{Av}_n(13\square 2)$.

Theorem 4.11. For all $n \geq 1$, $\pi = \alpha n\beta \in \mathrm{Av}_n(q)$ if and only if:

(i) $\alpha > \beta$, $\alpha,\beta \in \mathrm{Av}(q)$

(ii) $\alpha \not> \beta$, but $\alpha' > \beta'$, where $\alpha = \alpha' t_1$ and $\beta = t_2\beta'$ for some $t_1, t_2 \in [n-1]$. and one of the following holds:

1. $t_1 > \beta'$, $t_2 < \alpha'$, $t_1 < t_2$ and $\alpha',\beta' \in \mathrm{Av}(q)$.
2. $t_1 > \beta'$, $t_2 \not< \alpha'$, $\beta' \in \mathrm{Av}(q)$ and $\sigma = \alpha' t_1 t_2 \in \mathbb{H}_1$ with t_2 not being the smallest element in σ and not being the second smallest, after t_1.

3. $t_1 \not> \beta'$, $t_2 < \alpha'$, $\alpha' \in \mathrm{Av}(q)$ and $\sigma = t_1 t_2 \beta' \in \mathbb{H}_2$ with t_1 not being the biggest element in σ and not being the second biggest, after t_2.

4. $t_1 \not> \beta'$, $t_2 \not< \alpha'$, $\sigma_1 = \alpha' t_2 \in \mathbb{H}_1$ with t_2 not being the smallest element in σ_1 and $\beta' = x\beta''$, where $x > t_1 > \beta''$ and $\beta'' \in \mathrm{Av}(q)$.

The proof of this theorem is not included here. The described decomposition gives us the next result almost directly.

Theorem 4.12.

$$G(x) = 1 + G(x)(xH_1(x) + xH_2(x) + x^3 H_1(x)) + G^2(x)(x - 2x^2 - x^3 - x^4). \tag{4.9}$$

In order to obtain $G(x)$, we found a way to express $H_1(x)$ as a function of $H_2(x)$ and $G(x)$. Then we express $H_2(x)$ as a function of $G(x)$ using the block-decomposition method [115]. Extensive case analysis and inclusion-exclusion arguments are additionally used. As a result, we obtain a system of two equations each of which is a polynomial of x, $G(x)$ and $H_2(x)$. We eliminate $H_2(x)$ to obtain an equation $P(x, G(x)) = 0$, where P is a polynomial of $G(x)$ with coefficients that are polynomials of x. The polynomial P has 14 terms with the term of highest total degree being $x^8 G^6$. One could use a generalization of the Lagrange inversion formula discussed in the work of Baderier and Drmota [14] to get a closed-form expression for the coefficients of $G(x)$.

Recent work by Albert et al. [2] discusses a very general approach automating the discovery of similar decompositions of various sets of combinatorial objects. They claim that their computer method allowed them to find generating functions for the number of permutations avoiding other DPs of size 3 with two squares, namely $1\square 2\square 3$, $1\square^2 32$ and $13\square^2 2$.

4.4 Vincular distant patterns

In this section, we consider a particular kind of vincular distant patterns of size 3. The goal will be to establish a surprising fact related to the permutations avoiding the classical patterns 123 and 132.

4.4.1 Patterns of the form $\underline{ab}\square c$ and $a\square\underline{bc}$

There are 12 patterns of this kind, we have three symmetry classes and thus at most three Wilf-equivalence classes. We will show that these three symmetry classes are different, i.e., we have exactly three Wilf-equivalence classes.

Class 1	Class 2	Class 3
$\underline{12}\square 3$	$1\square\underline{32}$	$\underline{13}\square 2$
$\underline{32}\square 1$	$\underline{21}\square 3$	$\underline{31}\square 2$
$1\square\underline{23}$	$\underline{23}\square 1$	$2\square\underline{31}$
$3\square\underline{21}$	$3\square\underline{12}$	$2\square\underline{13}$

TABLE V: The Wilf-equivalent classes of $\underline{ab}\square c$ and $a\square\underline{bc}$ patterns.

We will first find a recurrence for the pattern $\underline{12}\square 3$.

Theorem 4.13. If $a_n = av_n(\underline{12}\square 3)$, then $a_n = n!$ for $0 \leq n \leq 3$, and for $n \geq 4$ we have

$$a_n = a_{n-1} + (n-1)a_{n-2} + \frac{(n+1)(n-2)}{2}a_{n-3} + \sum_{i=4}^{n-1}\left(\binom{n}{i-1} - 1\right)a_{n-i} + (n-1).$$

Proof. Let $q = \underline{12}\square 3$ and let $\pi = \pi_1\pi_2\cdots\pi_n$ be a permutation of $[n]$ that avoids q. We will consider five cases for the position of the number n in π. Denote this position with i, so $\pi_i = n$.

Case 1. $i = 1$: $\pi = n\pi_2\cdots\pi_n$

In this case, n will not participate in any occurrence of q since it can only be the first letter in such occurrence. Thus since π avoids q then $\pi_2\cdots\pi_n$ must avoid q. There are a_{n-1} such permutations $\pi_2\cdots\pi_n \in S_{n-1}$.

Case 2. $i = 2$: $\pi = \pi_1 n\cdots\pi_n$

Here, n cannot participate in any occurrence of q. Neither can π_1, because it could participate only together with $\pi_2 = n$. Then π_1 can be any of the remaining $n-1$ numbers. Regardless of the choice of π_1, one would have a_{n-2} ways to choose the order of the remaining $n-2$ letters since $\mathrm{red}(\pi_3\cdots\pi_n)$ must avoid q. This gives $(n-1)a_{n-2}$ ways to obtain π.

Case 3. $i = 3$: $\pi = \pi_1\pi_2 n\cdots\pi_n$

The number n cannot be part of a q-occurrence, again. Therefore if $n-1$ is in an occurrence of q, then it must be the last letter (the '3'). Let $j = \pi^{-1}(n-1)$ be the position of $n-1$ in π.

Case 3a. $j = 1$ or $j = 2$

None of the first three elements of π will be part of any occurrence of q. Thus we have $2(n-2)a_{n-3}$ permutations $\pi \in \mathrm{Av}_n(q)$ with $i = 3$ and $j = 1$ or $j = 2$, since we can choose the position, j, of $n-1$ in 2 ways and the other of the first 2 letters in $n-2$ ways. The rest of the permutation must avoid q and there are a_{n-3} possibilities for that. We get a q-avoiding permutation in all of these cases.

Case 3b. $j > 3$

Here, since π avoids q, we must have $\pi_1 > \pi_2$, because otherwise $\pi_1\pi_2\pi_j$ would be a q-occurrence. We can determine π_1 and π_2 in $\binom{n-2}{2}$ ways. The number of ways to determine $\pi_4 \cdots \pi_n$ would be again a_{n-3}, despite knowing that $n-1$ will be one of these letters, simply because this part of π must avoid q and because once we have π_1, π_2 and π_3 fixed, this part will correspond to a permutation in $\mathrm{Av}_{n-3}(q)$.

Case 4. $3 < i < n$. $\pi = \pi_1\pi_2 \cdots n \cdots \pi_n$

Since π avoids q, the numbers $\pi_1, \pi_2, \ldots \pi_{i-2}$ must be in decreasing order. We have three subcases for the position $j = \pi^{-1}(n-1)$.

Case 4a. $j = i-1$: $\pi = \pi_1\pi_2 \cdots \pi_{i-2}(n-1)n \cdots \pi_n$

The numbers $\pi_1, \ldots, \pi_{i-2}$ must be in decreasing order since $\pi \in \mathrm{Av}(q)$. Once we have chosen these $i-2$ numbers of π then neither $\pi_{i-1} = n-1$ nor $\pi_i = n$ could participate in a q-occurrence and any ordering of the last $n-i$ numbers that avoids q would give us a different q-avoider π. This gives $\binom{n-2}{i-2}a_{n-i}$ permutations for this case.

Case 4b. $j < i-1$ (in fact, $j = 1$): $\pi = (n-1)\pi_2 \cdots n\pi_{i+1} \cdots \pi_n$

This would imply that $j = 1$ since $\pi_1, \ldots, \pi_{i-2}$ are in decreasing order. If $\pi_{i-2} > \pi_{i-1}$, then we can select $\pi_2, \ldots, \pi_{i-1}$ in $\binom{n-2}{i-2}$ ways which gives $\binom{n-2}{i-2}a_{n-i}$ more q-avoiding permuta-

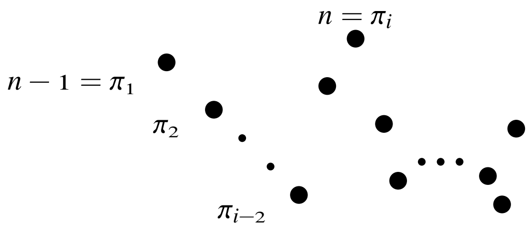


Figure 16: The order of the elements of π in Case $4b$.

tions. Slightly more attention is required for the subcase $\pi_{i-2} < \pi_{i-1}$. In order to avoid q, all of $\pi_{i+1}, \ldots, \pi_n$ must be smaller than π_{i-1}, because otherwise $\pi_{i-2}\pi_{i-1}\pi_k$ would be a q-occurrence for some $k > i$. Now, we should calculate how many different permutations π satisfy the described conditions. For clarity, one may look at Figure 16, which visualizes the order of the elements in one such π.

We claim that the number of these permutations is $\left(\binom{n-2}{i-3} - 1\right) a_{n-i}$. Indeed, we can first choose the last $n-i$ numbers $\pi_{i+1}, \pi_{i+2}, \ldots, \pi_n$, and the number π_{i-1}. Those are the unlabeled elements on Figure 16. We can do that in $\binom{n-2}{n-i+1} = \binom{n-2}{i-3}$ ways. Out of these choices, only the one where we have selected the smallest numbers, $1, 2, \ldots, n-i+1$, would force $\pi_{i-2} > \pi_{i-1}$ which we do not want to happen, so we exclude this single choice. For all the other choices, we simply have that the biggest number among the chosen has to be π_{i-1} and the other $n-i$ chosen numbers can be ordered in a_{n-i} ways at positions $i+1$, $i+2$, $\ldots$, n. The unchosen $i-3$ numbers are ordered decreasingly after $\pi_1 = n-1$, at positions $2, 3, \cdots, i-2$.

Case 4c. $j > i$

In this case, the numbers $\pi_1, \ldots, \pi_{i-1}$ must all be in decreasing order. Thus, it suffices just to choose which are they and choose the numbers in the remaining part of the permutation, i.e., we have $\binom{n-2}{i-1} a_{n-i}$ permutations here.

Case 5. $i = n$. $\pi = \pi_1 \pi_2 \pi_3 \cdots n$

Again, the numbers $\pi_1, \ldots, \pi_{n-2}$ must be in decreasing order, so it suffices to choose π_{n-1} in $n-1$ ways.

It remains to observe that in Case 4, after summing the number of q-avoiding permutations for the three subcases, we get

$$\left(\binom{n-2}{i-2} + \binom{n-2}{i-2} + \left(\binom{n-2}{i-3} - 1\right) + \binom{n-2}{i-1}\right) a_{n-i} =$$
$$= \left(\binom{n-2}{i-2} + \left(\binom{n-2}{i-3} - 1\right) + \binom{n-1}{i-1}\right) a_{n-i} =$$
$$= \left(\binom{n-1}{i-2} + \binom{n-1}{i-1} - 1\right) a_{n-i} = \left(\binom{n}{i-1} - 1\right) a_{n-i}.$$

□

The first few elements of the sequence $av_n(\underline{12}\square 3)$ for $n \geq 4$ are

$$20, 75, 316, 1464, 7359, 39815, 230306.$$

This is not part of any sequence in OEIS.

This enumerates avoidance for Class 1 patterns in Table Table V. Similar recurrence can be found for the patterns in Class 2. We give it below using the pattern $1\square\underline{32}$.

Theorem 4.14. If $b_n = av_n(1\square\underline{32})$, then $b_n = n!$ for $0 \leq n \leq 3$ and for $n \geq 4$ we have

$$b_n = b_{n-1} + (n-1)b_{n-2} + \frac{(n+1)(n-2)}{2}b_{n-3} + \sum_{i=2}^{n-3}\left(i\binom{n-2}{i} + \binom{n-1}{i-1}\right)b_{i-1} + (n-1). \quad (4.10)$$

The first few elements of the sequence $av_n(1\square\underline{32})$ for $n \geq 4$ are

$$20, 76, 326, 1544, 7954, 44164, 262456.$$

This is not part of any sequence in OEIS.

Theorems 4.13 and 4.14 differ only in the sums in their right-hand sides. Applying the complement map after the reverse map, we see that $av_n(\underline{12}\square 3) = av_n(1\square\underline{23})$ for every positive n and we already placed those two patterns in the same of the three classes for the considered set of vincular DPs. Using this, we can easily prove the following.

Theorem 4.15. If $n > 4$, then $av_n(1\square\underline{23}) < av_n(1\square\underline{32})$.

Proof. We just noted that $av_n(1\square\underline{23})$ is given by the number a_n from Theorem 4.13, while $av_n(1\square\underline{32})$ is given by the number b_n from Theorem 4.14. By substituting $j = n - i + 1$, we get that the sum in the right-hand side of Equation (4.10) can be written as

$$\sum_{j=4}^{n-1}((n-j+1)\binom{n-2}{j-3} + \binom{n-1}{j-1})b_{n-j}. \quad (4.11)$$

Then, in order to obtain this inequality, it suffices to prove that for every $n > 4$ and $4 \leq i \leq n-1$:

$$\binom{n}{i-1} - 1 < (n-i+1)\binom{n-2}{i-3} + \binom{n-1}{i-1}. \quad (4.12)$$

This is equivalent to $\binom{n-1}{i-2} - 1 < (n-i+1)\binom{n-2}{i-3}$ or $\binom{n-1}{i-2} - 1 < \frac{(n-i+1)(i-2)}{n-1}\binom{n-1}{i-2}$. When $n = 5$ and $i = 4$, we check directly that the latter holds. When $n > 5$, one can easily see that $\frac{(n-i+1)(i-2)}{n-1} > 1$, for $4 \leq i \leq n-1$. $\square$

It remains to investigate Class 3. A well known proof technique in the area of permutation patterns helps to do that.

Theorem 4.16. For all $n \in \mathbb{Z}^+$, $\mathrm{Av}_n(\underline{13}\square 2) = \mathrm{Av}_n(13\square 2)$, which implies that $av_n(\underline{13}\square 2) = av_n(13\square 2)$.

Proof. We will prove that whenever an n-permutation contains the pattern $13\square 2$, then it must contain the pattern $\underline{13}\square 2$. Take $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$ containing $q = 13\square 2$ and let $\sigma_i\sigma_j\sigma_k$, $1 \leq i < j < k-1 < n$ be an occurrence of q with the smallest possible distance between the 1 and the 3, i.e., $d = j - i$ is the smallest possible for such an occurrence. If $d = 1$, then $\sigma_i\sigma_j\sigma_k$ would be an occurrence of $\underline{13}\square 2$ and we are done. Assume that $d > 1$ and then consider the value of σ_{i+1}. If $\sigma_{i+1} < \sigma_k$, then $\sigma_{i+1}\sigma_j\sigma_k$ would be a q-occurrence with $j-(i+1) = d-1 < d$. On the other hand, if $\sigma_{i+1} > \sigma_k$, then $\sigma_i\sigma_{i+1}\sigma_k$ would be a q-occurrence with $(i+1) - i = 1 < d$, which is again a contradiction. $\square$

The theorem that we just proved and the fact that $av_n(12\square 3) = av_n(13\square 2)$ (see [70] and subsection 4.3.1) imply that $av_n(\underline{13}\square 2)$ is given by the right-hand side of Equation (4.6) and sequence A049124 in OEIS. It turns out that the patterns in the corresponding Class 3 of Table Table V have the smallest avoiding sets out of the 3 classes.

Theorem 4.17. For all $n \geq 5$, $av_n(\underline{12}\square 3) > av_n(\underline{13}\square 2)$.

To establish this fact, we will first need a few additional definitions. For a given pattern q, let $\mathrm{Av}_{i_1,i_2,\ldots,i_k;n}(q)$ be the set of q-avoiders of size n beginning with $i_1, i_2, \ldots, i_k$ and let $av_{i_1,i_2,\ldots,i_k;n}(q)$ denotes $av_{i_1,i_2,\ldots,i_k;n}(q)$. Let us first prove the following simple lemma.

Lemma 4.18. If $1 \le i \le n-1$ and $n \ge 4$, then

$$av_{i;n}(\underline{13}\square 2) \le av_{n;n}(\underline{13}\square 2) = av_{n-1}(\underline{13}\square 2).$$

Moreover, if $1 \le i \le n-2$, then the inequality is strict, i.e.,

$$av_{i;n}(\underline{13}\square 2) < av_{n;n}(\underline{13}\square 2).$$

Proof. For every $\pi \in \mathrm{Av}_{n-1}(\underline{13}\square 2)$, we have that $n\pi \in \mathrm{Av}_{n;n}(\underline{13}\square 2)$, since n cannot participate in any occurrences of $\underline{13}\square 2$, being at first position. Conversely, for every $n\sigma \in \mathrm{Av}_{n;n}(\underline{13}\square 2)$, one have that $\sigma \in \mathrm{Av}_{n-1}(\underline{13}\square 2)$. Thus, $av_{n;n}(\underline{13}\square 2) = av_{n-1}(\underline{13}\square 2)$. In addition, for every $1 \le i \le n-1$ and $i\sigma \in \mathrm{Av}_{i;n}(\underline{13}\square 2)$, we have $\mathrm{red}(\sigma) \in \mathrm{Av}_{n-1}(\underline{13}\square 2)$, which implies that $av_{i;n}(\underline{13}\square 2) \le av_{n-1}(\underline{13}\square 2)$.

Since $n \ge 4$, when $1 \le i \le n-2$, then we have at least one n-permutation $\pi = in\sigma'$, beginning with i, where $n\sigma' = \sigma$ is such that $\mathrm{red}(\sigma) \in \mathrm{Av}_{n-1}(\underline{13}\square 2)$ and i is obviously part of an $\underline{13}\square 2$-occurrence. An example is $\pi = ina \cdots (n-1)$ for any $a \in [n]$, where $a \ne i, n, (n-1)$. Therefore, $\mathrm{red}(\sigma) \in \mathrm{Av}_{n-1}(\underline{13}\square 2)$, but $\pi = i\sigma \notin \mathrm{Av}_{i,n}(\underline{13}\square 2)$. $\square$

We will need this lemma together with a few other definitions. Recall that $C_n(\sigma)$ denotes the permutations in S_n containing σ. Then, let

$$U_n := C_n(\underline{12}\square 3) \cap \mathrm{Av}_n(\underline{13}\square 2)$$

and

$$V_n := \mathrm{Av}_n(\underline{12}\square 3) \cap C_n(\underline{13}\square 2),$$

with $u_n := |U_n|$ and $v_n := |V_n|$. In addition, let us denote with $U_{i_1,i_2,\ldots,i_k;n}$ (resp. $V_{i_1,i_2,\ldots,i_k;n}$) the set of permutations in U_n (resp. in V_n) beginning with $i_1 i_2 \ldots i_k$. Furthermore, let $u_{i_1,i_2,\ldots,i_k;n} := |U_{i_1,i_2,\ldots,i_k;n}|$ and $v_{i_1,i_2,\ldots,i_k;n} := |V_{i_1,i_2,\ldots,i_k;n}|$. Now, we will prove the following.

Lemma 4.19. For each $n \geq 4$ and $1 \leq i \leq n$,

$$u_{i;n} \leq v_{i;n}.$$

Proof. Note that the statement implies $u_n \leq v_n$ and $|C_n(\underline{12}\square 3)| \leq |C_n(\underline{13}\square 2)|$ (resp., $av_n(\underline{12}\square 3) \geq av_n(\underline{13}\square 2)$), for each $n \geq 4$. Indeed, if $T_n = C_n(\underline{12}\square 3) \cap C_n(\underline{13}\square 2)$, then $C_n(\underline{12}\square 3) = U_n \cup T_n$ and $C_n(\underline{13}\square 2) = V_n \cup T_n$. Thus, $u_n \leq v_n$ implies $|C_n(\underline{12}\square 3)| \leq |C_n(\underline{13}\square 2)|$. We will proceed by induction on n. One can directly check that $u_{i;4} \leq v_{i;4}$ for each $1 \leq i \leq 4$. Now assume that $u_{i;n'} \leq v_{i;n'}$, for each $4 \leq n' \leq n-1$ and $1 \leq i \leq n'$, for a given $n \geq 5$. Consider $u_{i;n}$ and $v_{i;n}$ for $1 \leq i \leq n$. If $i = n$, then using the induction hypothesis, we have $u_{n;n} = u_{n-1} \leq v_{n-1} = v_{n;n}$. Similarly, if $i = n-1$, then we have $u_{n-1;n} = u_{n-1} \leq v_{n-1} = v_{n-1;n}$. Now, let $1 \leq i \leq n-2$. By the induction hypothesis, $u_{i,i-k;n} = u_{i-k;n-1} \leq v_{i-k;n-1} = v_{i,i-k;n}$, for each $1 \leq k \leq i-1$. It remains to compare the numbers $u_{i,i+k;n}$ and $v_{i,i+k;n}$ for $1 \leq k \leq n-i$. Note that when $k \geq 3$, then $u_{i,i+k;n} = 0$, since for these values of k, any n-permutation beginning with $i(i+k)$ will contain an occurrence of $\underline{13}\square 2$. Similarly, $v_{i,i+k;n} = 0$, when $i + k < n - 1$, since for these values of k, any n-permutation beginning with $i(i+k)$ will contain an occurrence of $\underline{12}\square 3$. We will show that $u_{i,i+1;n} \leq v_{i,n;n}$ and that $u_{i,i+2;n} \leq v_{i,n-1;n}$ which will complete the proof.

Consider the sets $U_{i,i+1;n}$ and $V_{i,n;n}$. First, let us look at those $\pi \in U_{i,i+1;n}$ (resp., $\pi \in V_{i,n;n}$) which do not begin with an $\underline{12}\square 3$ occurrence (resp., not with an $\underline{13}\square 2$ occurrence). Then, note that π must begin with $(n-2)(n-1)n$ (resp., with $(n-2)n(n-1)$). However, we have

$$u_{n-2,n-1,n;n} = u_{n-3} \leq v_{n-3} = v_{n-2,n;n-1} \tag{4.13}$$

using the induction hypothesis, again.

Now, let us look at those $\pi \in U_{i,i+1;n}$ beginning with a $\underline{12}\square 3$ occurrence. Their number is given by

$$av_{i;n-1}(\underline{13}\square 2) - av_{n-2,n-1;n-1}(\underline{13}\square 2). \tag{4.14}$$

Indeed, after we remove from π its first element i and flatten, we obtain an $(n-1)$-avoider of $\underline{13}\square 2$. Conversely, for any permutation $\pi = i\pi_2 \dots \pi_{n-1} \in \mathrm{Av}_{i;n-1}(\underline{13}\square 2)$, one can increase by 1 all the elements of π greater than or equal to i and then add i at the beginning, to obtain a permutation in $U_{i,i+1;n}$. This permutation will begin with a $\underline{12}\square 3$ occurrence, unless it begins with $(n-2)(n-1)n$, i.e., when $i = n-2$ and when $\pi \in \mathrm{Av}_{n-2,n-1;n-1}(\underline{13}\square 2)$. Therefore, we should subtract $av_{n-2,n-1;n-1}(\underline{13}\square 2)$. Respectively, for the number of permutations $\pi \in V_{i,n;n}$, beginning with an $\underline{13}\square 2$ occurrence, one would have

$$av_{n-1;n-1}(\underline{12}\square 3) - av_{n-1,n-2;n-1}(\underline{12}\square 3). \tag{4.15}$$

It is not difficult to see that $av_{n-2,n-1;n-1}(\underline{13}\square 2) = av_{n-3}(\underline{13}\square 2)$ and that $av_{n-1,n-2;n-1}(\underline{12}\square 3) = av_{n-3}(\underline{12}\square 3)$. Hence, by using expressions (4.14), (4.15) and (4.13), we see that in order to establish that $u_{i,i+1;n} \leq v_{i,n;n}$, it remains to prove the inequality below for each $1 \leq i \leq n-2$:

$$av_{i;n-1}(\underline{13}\square 2) - av_{n-3}(\underline{13}\square 2) \leq av_{n-1;n-1}(\underline{12}\square 3) - av_{n-3}(\underline{12}\square 3). \tag{4.16}$$

By Lemma 4.18, we have that $av_{i;n-1}(\underline{13}\square 2) \leq av_{n-1;n-1}(\underline{13}\square 2) = av_{n-2}(\underline{13}\square 2)$. We also have that $av_{n-1;n-1}(\underline{12}\square 3) = av_{n-2}(\underline{12}\square 3)$. Thus, it suffices to prove that

$$av_{n-2}(\underline{13}\square 2) - av_{n-3}(\underline{13}\square 2) \leq av_{n-2}(\underline{12}\square 3) - av_{n-3}(\underline{12}\square 3). \tag{4.17}$$

Using that $av_{n-3}(q) = av_{n-2;n-2}(q)$ for any of the patterns $q = \underline{12}\square 3$ or $q = \underline{13}\square 2$, as well as the relation

$$av_{i;n-2}(\underline{13}\square 2) \leq av_{i;n-2}(\underline{12}\square 3) \Longleftrightarrow v_{i;n-2} \geq u_{i;n-2}, \tag{4.18}$$

we see that Inequality (4.17) is equivalent to

$$\sum_{i=1}^{n-3} v_{i;n-2} \geq \sum_{i=1}^{n-3} u_{i;n-2}, \tag{4.19}$$

which follows directly, because by the induction hypothesis $u_{i;n-2} \leq v_{i;n-2}$, $\forall 1 \leq i \leq n-3$. From (4.13) and (4.19), we conclude that $u_{i,i+1;n} \leq v_{i,n;n}$.

One could establish that $u_{i,i+2;n} \leq v_{i,n-1;n}$ in almost the same way, by first noticing that $U_{i,i+2;n} = U_{i,i+2,i+1;n}$ and that $V_{i,n-1;n} = V_{i,n-1,n;n}$ since the permutations in $U_{i,i+2;n}$ (resp., in $V_{i,n-1;n}$) do not have a $\underline{13}\square 2$ (resp., a $\underline{12}\square 3$) occurrence. Then, the only thing that remains is to consider the cases $i = n-2$ and $i \neq n-2$ and to use the induction hypothesis and Lemma 4.18. In particular, if $i = n-2$, then

$$u_{n-2,n,n-1;n} = u_{n-3} \leq v_{n-3} = v_{n-2,n-1,n;n}. \tag{4.20}$$

If $i \neq n-2$, then $\pi \in U_{i,i+2,i+1;n}$ (resp., in $V_{i,n-1,n;n}$) begins with an $\underline{12}\square 3$ (resp., an $\underline{13}\square 2$) occurrence and

$$u_{i,i+2,i+1;n} = av_{i;n-2}(\underline{13}\square 2) \leq av_{n-2;n-2}(\underline{13}\square 2) \tag{4.21}$$

by Lemma 4.18. In addition,

$$av_{n-2;n-2}(\underline{13}\square 2) \leq av_{n-2;n-2}(\underline{12}\square 3) = v_{i,n-1,n;n} \tag{4.22}$$

by the induction hypothesis. □

As we have pointed out, Lemma 4.19 implies that $av_n(\underline{12}\square 3) \geq av_n(\underline{13}\square 2)$, for each $n \geq 4$. In order to obtain a proof of Theorem 4.17, we should just use the second part of Lemma 4.18 to see that Inequality (4.16) is strict when $n-1 \geq 4$, i.e., when $n \geq 5$.

The last Theorem 4.17 together with Theorem 4.15, the result on consecutive patterns of Elizalde [57] and the corollary of the result of Hopkins (Theorem 4.9) imply the following.

Corollary 4.20. Consider the set of distant patterns

$$X = \{1\square\underline{23}, \underline{12}\square 3, 1\square 2\square 3, \underline{123}\}.$$

Take any pattern $p \in X$ and switch the places of the letters 2 and 3 to get a pattern p' in $Y = \{1\square\underline{32}, \underline{13}\square 2, 1\square 3\square 2, \underline{132}\}$. We have that $\mathrm{Av}_n(p) > \mathrm{Av}_n(p')$ for all $n > 5$, $p \in X$ and the corresponding $p' \in Y$, except for $1\square\underline{23}$ which is avoided by fewer permutations of size n, compared to its counterpart $1\square\underline{32}$.

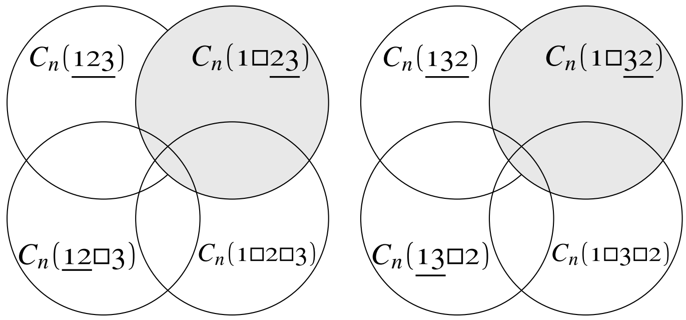


Figure 17: Venn diagrams for the n-permutations containing 123 and 132.

Figure 17 depicts the sets of permutations containing each of the patterns in X and Y as a Venn diagram. Corollary 4.20 is somewhat surprising since each occurrence of the classical pattern 123 (resp., 132) is an occurrence of a pattern in X (resp., Y) and as it was shown in [129], $av_n(123) = av_n(132)$. Thus the total "area" of the union of the four sets on the left is the same as the total "area" of the union of the four sets on the right. However, each of the three unmarked sets on the left contains fewer elements than its counterpart on the right.

4.4.2 Consecutive distant patterns

Recall that when all the constraints for the gap sizes in a distant pattern are tight, then we call these patterns consecutive distant patterns and we underline the whole pattern to denote that. Considering POGP, Kitaev mentioned in the introduction of [100] that $\mathrm{Av}_n(\underline{1\square 2}) = \binom{n}{\lfloor \frac{n}{2} \rfloor}$. Indeed, we may use that the letters in the odd and the even positions of a permutation avoiding this pattern do not affect each other. Thus we can choose the letters in odd positions in $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ ways, and we must arrange them in decreasing order. We then must arrange the letters in even positions in decreasing order, too. Using the same reasoning one can easily find, for example $\mathrm{Av}_n(\underline{1\square^2 2})$ or $\mathrm{Av}_n(\underline{1\square 2\square 3})$. This can be further generalized by the fact given below. Recall that if $q = q_1q_2\cdots q_k$ is a classical pattern of size k, then $\underline{q} = \underline{q_1q_2\cdots q_k}$ is the corresponding consecutive pattern. We also use $\underline{\mathrm{dist}_r(q)}$ to denote the corresponding consecutive distant pattern $\underline{q_1\square^r q_2\square^r \cdots \square^r q_k}$.

Theorem 4.21. [99, Theorem 11] For a given classical pattern q of size k, given distance $r \geq 0$ and a natural n, denote $l = \lfloor \frac{n}{r+1} \rfloor$. Set $u := n \mod (r+1) \in [0, r]$. Then

$$av_n(\underline{\mathrm{dist}_r(q)}) = \frac{n!}{(l!)^{r+1-u}((l+1)!)^u} |A_l(\underline{q})|^{r+1-u} |A_{l+1}(\underline{q})|^u. \tag{4.23}$$

This gives us a formula for the size of the set of permutations avoiding any consecutive distant pattern, knowing the size of the avoidance set for the corresponding classical consecutive pattern. Corollaries of this simple fact were previously stated in [70, 71]. We state another simple corollary here, which shows a surprising relationship between the last Theorem 4.21 and avoidance of arithmetic progressions in permutations.

Theorem 4.22. The number of permutations of size n avoiding arithmetic progressions of length $k > 1$ and difference $r > 0$ is $av_n(\underline{\mathrm{dist}_r(12\cdots k)})$, which can be obtained using Equation (4.23).

Proof. Consider $\pi \in S_n$, containing an arithmetic progression $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}$ of size k and difference $r > 0$. I.e., we have $\pi_{i_1} = x, \pi_{i_2} = x+r, \cdots, \pi_{i_k} = x+(k-1)r$ for some $x, r \in [n]$ with $i_1 < i_2 < \cdots < i_k$. Then in the inverse permutation π^{-1}, $i_1 i_2 \cdots i_k$ will be an occurrence of the distant pattern $\underline{\mathrm{dist}_r(12\cdots k)}$ since $\pi^{-1}(x) = i_1, \pi^{-1}(x+r) = i_2, \cdots, \pi^{-1}(x+(k-1)r) = i_k$. Conversely, if $\pi \in S_n$ contains $\underline{\mathrm{dist}_r(12\cdots k)}$, then π^{-1} contains an arithmetic progression of length k and difference $r > 0$. Therefore, the number of permutations of $[n]$ containing arithmetic progressions of length k and difference $r > 0$ equals the number of permutations of $[n]$ containing $\underline{\mathrm{dist}_r(12\cdots k)}$. This implies the same for the set of avoiders, i.e., what we aim to prove. $\square$

4.5 Interpretations of other results with DPs

Here, we will demonstrate that DPs can be very useful when interpreting already known results (including ones obtained with a computer). One previous work that gives several conjectures about the enumeration of pattern-avoiding classes containing many size four patterns is the work of Kuszmaul [105]. He listed ten conjectures about simultaneous pattern avoidance of many size four patterns. One can find brief solutions, using both computer programs and manual work, to many of these conjectures in the two articles of Mansour and Schork [112, 113].

Below, we give direct solutions to two of the conjectures without using a computer. To do that, we interpret the respective big set of size four patterns as a smaller set of both classical and distant patterns. Our approach is similar to the technique introduced in [114].

Theorem 4.23. (conjectured in [105, p.20, sequence 6]) The generating function of

$$av_n(2431, 2143, 3142, 4132, 1432, 1342, 1324, 1423, 1243)$$

is given by $C + x^3C$, where C is the generating function for the Catalan numbers.

Proof. Note that the set of patterns above can be written as

$$\Pi = \{\square 132, 132\square, 1342\}$$

When $n = 1, 2, 3$, $av_n(\Pi) = 1, 2, 6$ respectively and these are indeed the first three coefficients of $C + x^3C$. Consider values $n \geq 4$. If $\sigma \in \mathrm{Av}_n(132)$, then $\sigma \in \mathrm{Av}_n(\Pi)$. As we know, $av_n(132)$ has generating function C [104]. It remains to find the generating function for those σ containing 132, but avoiding Π. Take one such $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ and notice that any occurrence of 132 in σ must have σ_1 as its first letter and σ_n as its last letter. Otherwise, given that $n \geq 4$, an occurrence of at least one of the patterns $132\square$ or $\square 132$ will be present. Now, let $\sigma_k = n$ be the biggest element of σ. Clearly, $\sigma_1\sigma_k\sigma_n$ must be an occurrence of 132. If not, then this biggest element must be either at the first or the last position in σ and thus σ would not contain any 132-occurrences that either begin at σ_1 or end at σ_n. In Figure 18 are shown three black points representing σ_1, σ_k and σ_n, as well as three segments of the diagram of σ denoted with A, B and C and defined below. We further show that σ will not contain any elements in these three segments. Here is why:

- A is empty

  There is no element x among $\sigma_2, \sigma_3, \cdots, \sigma_{k-1}$, such that $x < \sigma_n$. If there is such x, then $x\sigma_k\sigma_n$ would be a forbidden occurrence of 132.

- B is empty

  There is no element x among $\sigma_{k+1}, \sigma_{k+2}, \cdots, \sigma_{n-1}$, such that $\sigma_1 < x < \sigma_k$. If there is such x, then $\sigma_1\sigma_k x$ would be a forbidden occurrence of 132.

- C is empty

  There is no element x among $\sigma_1, \sigma_2, \cdots, \sigma_{k-1}$, such that $\sigma_n < x < \sigma_k$. If there is such x, then $\sigma_1 x\sigma_k\sigma_n$ would be an occurrence of 1342.

Therefore, the biggest element σ_k in σ must be at position 2, i.e., $k = 2$ and the only non-empty segment could be the one denoted by α in figure Figure 18. In other words, we must have $\sigma = (n-2)n\alpha(n-1)$, for some permutation $\alpha \in \text{Av}(132)$. Otherwise, an occurrence of 132, such that $\sigma_1 = n-2$ is not part of it, would be formed. Conversely, for any $\alpha \in \text{Av}(132)$, $\sigma = (n-2)n\alpha(n-1)$ belongs to $\text{Av}(\Pi)$.

Then, we get x^3C for the generating function of the permutations in $\text{Av}(\Pi)$ containing 132 and therefore we will have $C + x^3C$ for the generating function of $\text{Av}(\Pi)$, since C is the generating function for $\text{Av}(132)$. □

As we know, $C = 1 + xC^2$, so we can write

$$C + x^3C = C + x^3(1 + xC^2) = x^3 + C(1 + x^4C)$$

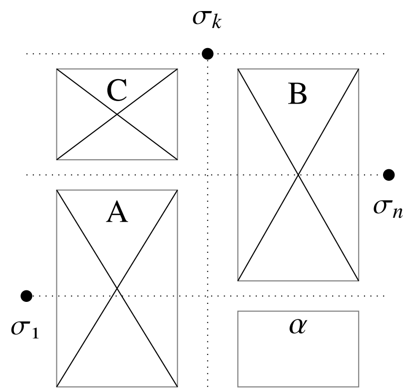


Figure 18: Decomposition for $\sigma \in \mathrm{Av}(\Pi)$ from the proof of Theorem 4.23.

and this indeed corresponds to sequence [118, A071742] given by $C(1+x^4C)$, as reported in [105], with the subtle difference that for $n = 3$, we have one extra permutation in $\mathrm{Av}_3(\Pi)$, namely 132. The same structure for the decomposition of the permutations in $\mathrm{Av}(\Pi)$ was also found recently with a computer by Bean et al., who used a particular algorithm called the *Struct algorithm* [17]. As we saw, rewriting the problem in terms of DPs helped us to prove the result directly and to give an interpretation of the already discovered decomposition.

Below, we will give a direct proof to another former conjecture listed in [105].

Theorem 4.24. (conjectured in [105, p.19, sequence 5]) The generating function of

$$av_n(2431, 2413, 3142, 4132, 1432, 1342, 1324, 1423)$$

is given by $C(1 + x^3C)$, where C is the generating function for the Catalan numbers.

Proof. Note that the set of permutations above can be written as

$$\Pi = \{13\square 2, 1324, 2431, 3142, 4132\}.$$

If σ has no occurrences of 132 at all, then obviously $\sigma \in \mathrm{Av}(\Pi)$ and the generating function for these permutations is C. Let us consider those σ that have some occurrences of 132 and are in $\mathrm{Av}(\Pi)$. The set Π contains $13\square 2$ thus all the occurrences of 132 in σ, are occurrences of $1\underline{32}$. Take the occurrence $\sigma_i\sigma_j\sigma_{j+1}$ of $1\underline{32}$ that ends at the largest possible position, i.e., with j maximal. Denote by α the segment in σ of largest possible size that ends at σ_i and such that $\alpha < \sigma_{j+1} < \sigma_j$. Let us first consider $\sigma' = \sigma_{j+2}\sigma_{j+3}\cdots\sigma_n$. We will show that σ' is the empty permutation, i.e., $n = j+1$ and the segments A,B and C, defined below and shown at Figure 19 are empty.

- A is empty

  There is no element x in σ', such that $x < \sigma_j$ and $x \nless \alpha$. If there is such x, then $\sigma_i\sigma_j x$ would be an occurrence of 132 that is not an $1\underline{32}$—occurrence.

- B is empty

  There is no element x in σ', such that $x < \alpha$. If there is such x, then $\sigma_i\sigma_j\sigma_{j+1}x$ would be an occurrence of 2431 which is not allowed.

- C is empty

  There is no element x in σ', such that $x > \sigma_j$. If there is such x, then $\sigma_i\sigma_j\sigma_{j+1}x$ would be an occurrence of 1324 which is not allowed.

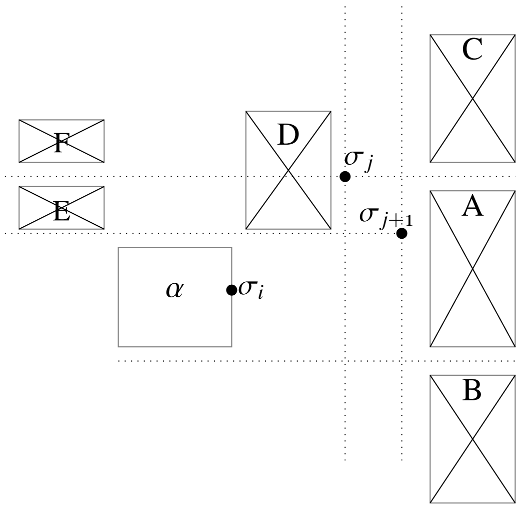


Figure 19: Decomposition for $\sigma \in \mathrm{Av}(\Pi)$ from the proof of Theorem 4.24.

Next, let us consider the segment $\sigma'' = \sigma_{i+1} \cdots \sigma_{j-1}$. We will show that σ'' is the empty permutation, i.e., the segment D, shown at Figure 19, is empty and $i = j - 1$.

- D is empty

  There is no element x in σ'', such that $x > \sigma_{j+1}$. If there is such x, then $\sigma_i x \sigma_{j+1}$ would be an occurrence of 132 that is not an $1\underline{32}$—occurrence.

Finally, consider the segment σ''' that is the part of σ in front of α. We will show that σ''' is the empty permutation, i.e., the segments E and F, shown at Figure 19 are empty.

- E is empty

  There is no element x in σ''', such that $x > \sigma_{j+1}$ and $x < \sigma_j$. If there is such x, then $x\sigma_i\sigma_j\sigma_{j+1}$ would be an occurrence of 3142, which is forbidden.

- F is empty

  There is no element x in σ''', such that $x > \sigma_j$. If there is such x, then $x\sigma_i\sigma_j\sigma_{j+1}$ would be an occurrence of 4132, which is forbidden.

Therefore, we must have $\sigma = \alpha n(n-1)$, where α is non-empty and $\alpha \in \text{Av}(132)$. The latter holds since if α contains 132 then after appending σ_j at the end, we will get an occurrence of 1324. Conversely, one may readily check that for each non-empty $\alpha \in \text{Av}(132)$, $\sigma = \alpha n(n-1)$ would be a permutation in $\text{Av}(\Pi)$. Thus, the generating function of the number of permutations in $\text{Av}(\Pi)$ is $C + x^2(C-1)$, since the generating function of such non-empty $\alpha \in \text{Av}(132)$ is $C-1$. Furthermore, we have

$$C + x^2(C-1) = C + x^2 xC^2 = C(1 + x^3 C).$$

□

The sequence of the coefficients for this generating function is given by A071726 in OEIS.

4.6 Stanley–Wilf type conjectures for DPs

In this section, we will consider some analogues of the Stanley-Wilf conjecture for DPs. The required definitions and the historical background on the Stanely-Wilf conjecture for classical patterns can be found in Section 1.1.3.

In Theorem 4.1, we saw that the avoidance of every distant pattern is equivalent to simultaneous avoidance of several classical patterns. The Stanley–Wilf conjecture is true for any of these classical patterns. Thus we will have that $\sqrt[n]{av_n(q)} < const$, for any distant pattern q, when $n \to \infty$. Arratia's observation that $av_{m+n}(q) \geq av_m(q).av_n(q)$, also holds for distant patterns, if the considered distant pattern does not start with a square. Thus for those kind of DPs, we can rely on the Fekete's lemma on subadditive sequences, exactly as Arratia did, to obtain that $\sqrt[n]{av_n(dp)}$ exists. As for the DPs beginning with $r > 0$ number of squares, we can use Theorem 4.2 to write

$$\sqrt[n]{av_n(\Box^r q)} = \sqrt[n]{n^{(r)} av_{n-r}(q)} = (n^{(r)})^{\frac{1}{n}} av_{n-r}(q)^{\frac{1}{n-r}\frac{n-r}{n}} \underset{n\to\infty}{\longrightarrow} c_q,$$

where q is a distant pattern which does not start with a square and $\sqrt[n]{av_n(q)} \longrightarrow c_q$. This yields the following Stanley–Wilf type result for DPs.

Theorem 4.25. For any distant pattern q, there exists a constant $c > 0$, such that

$$\sqrt[n]{av_n(q)} \underset{n\to\infty}{\longrightarrow} c_q. \tag{4.24}$$

An interesting continuation might be to consider avoidance of $\mathrm{dist}_r(q)$, for a classical pattern q and size of r that increases with n. Obviously, if $r \geq n-1$, then $av_n(\mathrm{dist}_r(q)) = n!$ for any pattern q. However, one may ask what will happen if r is a fixed positive fraction of n. Is it true that when n is growing, the number of permutations avoiding the corresponding series of DPs is still always converging to c^n, for some constant c? Below, we show that this is not the case, by using Theorem 4.4.

Theorem 4.26. It is not true that for any given classical pattern q, there exist constants $c > 0$ and $0 < c_1 < 1$, such that

$$\sqrt[n]{av_n(\mathrm{dist}_r(q))} \xrightarrow[r=\lfloor c_1 n\rfloor]{n\to\infty} c \tag{4.25}$$

Proof. Consider the classical pattern $q = 12$. By Theorem 4.4, the number of n-permutations avoiding $1\square^r 2$, for any $r \geq 1$, will be the same as the number of n-permutations for which in each of their cycles, any two elements differ by at most r. Denote this set of permutations by S_n^r. Furthermore, let C_n^r be the set of permutations in S_n for which each cycle is of length exactly r, except possibly 1 cycle of smaller length, if r does not divide n, and where each cycle is consisted of consecutive elements. Therefore, since $C_n^r \subseteq S_n^r$, we can use that $|C_n^r| \leq |S_n^r| = av_n(1\square^r 2)$. In addition, we have the obvious bound $|C_n^r| \geq ((r-1)!)^{\lfloor \frac{n}{r} \rfloor}$. Thus for any given $0 < c_1 < 1$, if $r = \lfloor c_1 n \rfloor$, then for big enough values of n, we have:

$$\mathrm{Av}_n(1\square^r 2) \geq ((r-1)!)^{\lfloor \frac{n}{r} \rfloor} = ((\lfloor c_1 n \rfloor - 1)!)^{\lfloor \frac{n}{\lfloor c_1 n \rfloor} \rfloor} = ((\lfloor c_1 n \rfloor - 1)!)^{\frac{1}{c_1}} \geq$$

$$((\frac{c_1}{2} n)!)^{\frac{1}{c_1}} \geq ((\frac{c_1 n}{2e})^{\frac{c_1}{2} n})^{\frac{1}{c_1}} = (\frac{c_1 n}{2e})^{\frac{n}{2}} = \sqrt{C n^n} = \Omega(C^n),$$

for some constant $C > 0$. In the last equation, we used the Stirling approximation. □

The latter fact motivates us to consider the following conjecture.

Conjecture 4.27. For any given classical pattern q and for every $0 < c_1 < 1$, there exists $0 < w < 1$, such that:

$$\lim_{n \to \infty} \left(\frac{av_n(\mathrm{dist}_{\lfloor c_1 n \rfloor}(q))}{n!} \right)^{\frac{1}{n}} = w \tag{4.26}$$

The approach of Elizalde ([55, Section 4]) for consecutive patterns might be useful when one tries to prove the latter conjecture, even though this approach cannot be applied directly. Here, we prove one lemma that might help confirming the conjecture.

Lemma 4.28. For any given classical pattern $q \in S_k$ and for every $0 < c_1 < 1$, there exists $d < 1$, such that if $r = \lfloor c_1 n \rfloor$ and $n \geq k(r+1)$, then

$$av_n(\mathrm{dist}_r(q)) < d^n n!. \tag{4.27}$$

Proof. Assume that $n \geq k(r+1)$ for some c_1 and n and let us take an arbitrary permutation $\pi = \pi_1\pi_2\cdots\pi_n$. We can divide the elements of π into roughly $\frac{n}{k}$ non-overlapping subsequences of size k, such that if $\pi \in \mathrm{Av}_n(\mathrm{dist}_r(q))$, then neither of these subsequences is order-isomorphic to q. We are looking for an upper bound and thus such a necessary condition could help. One way to get such a partition into subsequences is to take

$$\{\pi_1\pi_{r+2}\cdots\pi_{(k-1)r+k}, \pi_2\pi_{r+3}\pi_{2r+4}\cdots, \pi_{r+1}\pi_{2r+2}\cdots\},$$

with the first element in every next subsequence being the first not yet used element of π. Denote this family of subsequences by $\mathbb{F}$ and the event that after a uniform sampling of a permutation π, no subsequence in $\mathbb{F}$ is order isomorphic to q by $E_{\mathbb{F},q}$. Since $|\mathbb{F}| \geq (r+1) > 0$, we will have that

$$\mathbb{P}(E_{\mathbb{F},q}) \leq \left(1 - \frac{1}{k!}\right)^{r+1}$$

Therefore, if we write $C_{\pi,\mathrm{dist}_r(q)}$ for the event that π contains the pattern $\mathrm{dist}_r(q)$, then

$$\mathbb{P}(C_{\pi,\mathrm{dist}_r(q)}) > 1 - \left(1 - \frac{1}{k!}\right)^{r+1}$$

and thus the number of permutations π containing $\mathrm{dist}_r(q)$ is at least $n!(1-\left(1-\frac{1}{k!}\right)^{r+1})$ from which we can deduce that $av_n(\mathrm{dist}_r(q)) \leq n!\left(\left(1-\frac{1}{k!}\right)^{r+1}\right) = \left(\left(1-\frac{1}{k!}\right)^{\frac{r+1}{n}}\right)^n n! = d^n n!$, for $d = \left(1-\frac{1}{k!}\right)^{\frac{r+1}{n}}$. □

An analogous fact could be conjectured about the bound from below, which would lead to a proof of Conjecture 4.27, since we have Lemma 4.28.

Conjecture 4.29. For any given classical pattern $q \in S_k$ and for every $0 < c_1 < 1$, there exists $c > 0$, such that if $r = \lfloor c_1 n \rfloor$ and $n \geq k(r+1)$, then

$$av_n(\mathrm{dist}_r(q)) > c^n n!. \tag{4.28}$$

We saw that when r is a positive fraction of n, the number of n-permutations avoiding the corresponding distant pattern may become huge. Thus it would be reasonable to consider a Stanley–Wilf type conjecture, where r is asymptotically smaller than $O(n)$, e.g., a function of the kind n^{c_2}, for $0 < c_2 < 1$.

Conjecture 4.30. For any given classical pattern q, there exist $c_q > 0$ and $0 < c_2 < 1$, such that

$$\sqrt[n]{av_n(\mathrm{dist}_r(q))} \xrightarrow[r=\lfloor n^{c_2} \rfloor]{n\to\infty} c_q \tag{4.29}$$

If the latter conjecture is true, then one might ask which are the allowable growth rates c_q when c_2 is a fixed positive constant. Furthermore, an interesting additional question would be to find a function $g(n)$, such that

$$\sqrt[n]{av_n(\mathrm{dist}_r(q))} \xrightarrow[r=\lfloor \Theta(g(n)) \rfloor]{n\to\infty} c,$$

for some constant $c > 0$, but

$$\sqrt[n]{av_n(\mathrm{dist}_r(q))} \underset{r=\lfloor\Omega(g(n))\rfloor}{\overset{n\to\infty}{\not\longrightarrow}} c,$$

for any $c > 0$.

CHAPTER 5

FURTHER QUESTIONS

5.1 Sorting and shuffling

The sorting devices considered in Chapter 2 and the obtained results raise some additional questions.

1. Can we use Theorem 2.6 to make progress on the long-standing problem of finding the number of permutations sortable by a deque [118, A182216]? Some results on the asymptotic of these numbers can be found in [124, 125].

2. Can we find shuffle queues that are equivalent to the input and the output restricted deques defined in Section 1.3.3? In general, if T is a set of patterns, then for which T exists a shuffle queue $\mathbb{Q}_\Sigma$, such that $S_n(\mathbb{Q}_\Sigma) = Av_n(T)$, for each $n \geq 2$?

3. Find characterizations in terms of pattern avoiding classes for the set of permutations of given cost. Theorem 2.9 gives such a characterization for the set of permutations of cost 1.

4. Is it true that

$$M(n) \xrightarrow[n\to\infty]{} \lceil \log_2 n \rceil ?,$$

where $M(n)$ is the maximal cost of a permutation of size n, defined after Theorem 2.14. A positive answer to this question would imply that every permutation of size n can be sorted using $O(n \log n)$ operations by using cuts since one iteration uses $O(n)$ operations.

5. In Section 2.1.2, we noted that there exists a deterministic linear time algorithm that sorts all of the permutations in $S_n(\mathbb{Q}'_{\text{cuts}})$. Which are the shuffling methods Σ for which there exists such a linear procedure that sorts all of the permutations in $S_n(\mathbb{Q}'_{\Sigma})$?

6. Find characterization of the shuffling methods, whose shuffle queues without restrictions or of types (i) and (ii), can sort all permutations in S_n? Theorem 2.20 identifies one class of such shuffling methods for shuffle queues of type (ii).

5.2 Moments of permutation statistics

Here, we discuss four interesting further questions related to the results in Chapter 3.

1. Can we improve the bounds for the number of terms in Equation (3.2) and for the number of terms in Equation (3.7)? Some computational evidence suggests that this might be possible.

2. Can we prove the central limit theorem for vincular patterns by giving either a combinatorial or algebraic proof to Equation (3.13) in Theorem 3.26?

3. Which are the possible asymptotic distributions of $\text{cnt}_{\underline{P}}$ for other bivincular patterns, except $(21, \{1\}, \{1\})$, which was shown to be Poisson in Section 3.4.3? This question has been already stated in [84, Section 1], where some approaches were also suggested.

4. Theorem 3.10 shows that the aggregate of any permutation statistic is a linear combination of shifted factorials with constant coefficients. Similarly, in [96], Khare et al. showed that any statistic on matchings is a linear combination of double factorials with constant coefficients, whereas for statistics on the more general structure of set partitions, Chern et al. [37] showed that we have linear combinations of shifted Bell numbers with polynomial coefficients. These facts suggest that

most probably, there exists a combinatorial structure generalizing permutations, for which the aggregates of the statistics on it can be written as linear combinations of factorials with polynomial coefficients. Can we find such a structure, e.g., posets or polyominoes?

5.3 Distant patterns

Some ideas for further investigations, related to distant patterns and chapter 4 are listed below.

1. The following surprising conjecture was formulated with the help of a computer.

   Conjecture 5.1. Choose one of the 3 places between consecutive letters in the patterns $\{1234, 1243, 2143\}$ and put a square at that place for each of the three given classical patterns. You will obtain three Wilf-equivalent distant patterns. For example,

   $$av_n(1\square 234) = av_n(1\square 243) = av_n(2\square 143). \tag{5.1}$$

   To the best of our knowledge, none of the parts of this conjecture has been already resolved or follows from previous results. We should note that a similar statement does not hold for any two Wilf-equivalent classical patterns, because $av_n(4132) = av_n(3142)$ [131], for all $n > 1$, but $|A_7(4\square 132)| = 3592 \neq 3587 = |A_7(3\square 142)|$.

2. Uniform distant patterns are discussed in Section 4.3. Below, we list three conjectures related to the least and most avoided uniform distant pattern:

Conjecture 5.2. For every $m \geq 3$ and $r \geq 1$, there exists $n_0 \in \mathbb{N}$ such that for every natural $n > n_0$, we have

$$av_n(\mathrm{dist}_r(12\cdots m)) \geq av_n(\mathrm{dist}_r(q)), \tag{5.2}$$

for any given classical pattern q of size m.

Conjecture 5.3. For every $m \geq 3$ and $r \geq 1$, there exists $n_0 \in \mathbb{N}$ such that for every natural $n > n_0$, we have

$$av_n(\mathrm{dist}_r(12\cdots(m-2)m(m-1))) \leq av_n(\mathrm{dist}_r(q)), \tag{5.3}$$

for any given classical pattern q of size m.

A weaker version of these two conjectures would be the one below and a suitable injection establishing the fact is desired.

Conjecture 5.4. For every $m \geq 3$ and $r \geq 1$, there exists $n_0 \in \mathbb{N}$ such that for every natural $n > n_0$:

$$av_n(\mathrm{dist}_r(12\cdots m)) \geq av_n(\mathrm{dist}_r(12\cdots(m-2)m(m-1))). \tag{5.4}$$

3. Three conjectures related to Stanley–Wilf type results for distant patterns, namely Conjecture 4.27, 4.29 and 4.30, are listed in Section 4.6.
4. In addition to distant pattern, one may consider *bivincular distant patterns*, i.e., vincular distant patterns with constraints for the values of the letters in each of their occurrences.

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